

HIRZEBRUCH-RIEMANN-ROCH THEOREM FOR DG ALGEBRAS

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Dedicated to the memory of L. L. Vaksman

1. INTRODUCTION

1.1. Geometry of DG categories. To motivate the subject of the present research, we will begin by discussing some applications of triangulated and differential graded categories in algebraic geometry.

Let X be a quasi-compact separated scheme.¹ Denote by $D_{\text{qcoh}}(X)$ the derived category of complexes of \mathcal{O}_X -modules with quasi-coherent cohomology and by $D_{\text{perf}}(X)$ its triangulated subcategory of perfect complexes, i.e. complexes which are locally quasi-isomorphic to finite complexes of vector bundles. The category $D_{\text{perf}}(X)$ has proved to be the basic (co)homological invariant of X which somehow encodes all other reasonable invariants. This idea underlies R. Thomason's research on the K -theory of schemes [64], M. Kontsevich's Homological Mirror Symmetry program [39], and A. Bondal's and D. Orlov's research on the derived categories of smooth schemes [9].²

When working with $D_{\text{perf}}(X)$, one faces the following problem: even though various invariants of X depend on this category, it is not clear how to compute some of them in terms of $D_{\text{perf}}(X)$, viewed as an abstract triangulated category. One way to get around the problem is due to A. Bondal and M. Kapranov [7]. The point is that the derived categories, as opposed to abstract triangulated categories, can be “upgraded” to differential graded (DG) categories. In practice this can be achieved by, say, passing from $D_{\text{perf}}(X)$ to the DG category $\text{Perf}X$ of left bounded injective perfect complexes. The category $D_{\text{perf}}(X)$ is then recovered as the homotopy category of $\text{Perf}X$. Many other invariants of X can be extracted from $\text{Perf}X$ as well. The simplest example is the computation of the Hodge cohomology of X in terms of $\text{Perf}X$ in the case of a smooth scheme. One has

$$(1.1) \quad \text{HH}_n(\text{Perf}X) = \bigoplus_i \text{H}^{i-n}(\Omega_X^i),$$

where the left-hand side stands for the n -th Hochschild homology group of $\text{Perf}X$ (see Section 2.3). The category $\text{Perf}X$ encodes also some geometric properties of X . For example, if X is smooth then the category $\text{Perf}X$ is a perfect bimodule over itself [41].

The DG categories of the form $\text{Perf}X$ turn out to be equivalent to the DG categories of perfect modules over certain DG algebras. Namely, according to [10, Section 3.1] (see also [57]) $D_{\text{perf}}(X)$ is generated by a single perfect complex, \mathcal{E} . Let $A = \text{End}_{\text{Perf}X}(\mathcal{E})$.

¹In what follows, everything is considered over a fixed ground field.

²One of their results claims that schemes of certain type can be completely reconstructed from their derived categories.

Then $\text{Perf } X$ is quasi-equivalent to the DG category $\text{Perf } A$ of perfect right A -modules (see Section 2.2 for the definition of the latter category). Of course, there is no canonical generator of $D_{\text{perf}}(X)$ and, as a result, there is no canonical DG algebra associated with the scheme. However any DG algebra such that $\text{Perf } X$ is quasi-equivalent to $\text{Perf } A$ can be viewed as a replacement of the algebra of regular functions in the case of a non-affine scheme X .

Let us look at the most popular example - the projective line \mathbf{P}^1 . Due to the well known result of A. Beilinson [3], the derived category of coherent sheaves in this case is equivalent to the derived category of finite dimensional modules over the path algebra of the Kronecker quiver:



Following [65], we will say that two DG algebras A and B are Morita-equivalent if their perfect categories $\text{Perf } A$ and $\text{Perf } B$ are quasi-equivalent. In view of the above discussion, each scheme gives rise to a fixed Morita-equivalence class. Therefore it is reasonable to think of an *arbitrary* Morita-equivalence class as representing some noncommutative scheme or, better yet, a *noncommutative DG-scheme*. Any DG algebra from the equivalence class should be viewed as “the” algebra of regular functions on this noncommutative DG-scheme, and $\text{Perf } A$ plays the role of $\text{Perf } X$.

The above point of view agrees with the philosophy of *derived noncommutative algebraic geometry*.³ This subject was initiated in the beginning of 90’s based on the previous extensive study of derived categories of coherent sheaves undertaken by the Moscow school (A. Beilinson, A. Bondal, M. Kapranov, D. Orlov, A. Rudakov et al). Later on, it was greatly enriched by new ideas and examples coming from M. Kontsevich’s Homological Mirror Symmetry program [38]. A particularly important implication of the program is that one can associate certain triangulated categories with symplectic manifolds which should play the same important role in symplectic geometry that the derived categories of coherent sheaves play in algebraic geometry. Further important ideas and results in the field are due to A. Bondal and M. Van den Bergh, T. Bridgeland, V. Drinfeld, B. Keller, M. Kontsevich and Y. Soibelman, D. Orlov, R. Rouquier, B. Toen and others.

Of course, a “real” definition of noncommutative DG-schemes should include also a description of morphisms between them. It is clear that morphisms are given by DG functors between the categories of perfect complexes (a prototype is the pull-back functor associated with a morphism of schemes). The real definition is more subtle and we won’t discuss it here referring the reader to more thorough treatments of the subject [20, 35, 62, 63, 65, 66].

Here is an interesting question: Is it possible to tell whether a noncommutative DG-scheme comes from a usual commutative one? There is a simple necessary condition: the corresponding DG algebra A should be Morita-equivalent to its opposite DG algebra A^{op} (the simplest case when this is so is when the DG algebras A and A^{op} are isomorphic; look at the Kronecker quiver!). Of course, this condition is not sufficient: various almost commutative schemes, such as orbifolds, also satisfy it.

³We are not sure whether this name is commonly accepted or not.

Let A be a DG algebra. Then, following [41], one can define the corresponding non-commutative DG-scheme to be

proper iff so is A , i.e. $\sum_n \dim H^n(A) < \infty$;

smooth iff so is A , i.e. A is quasi-isomorphic to a perfect A -bimodule.

The first property is central to the present work, although we will touch upon smooth DG algebras as well (see Section 6).

1.2. A categorical version of the Hirzebruch-Riemann-Roch theorem. Let us turn now to the subject of this article, Hirzebruch-Riemann-Roch (HRR) theorem in the above noncommutative setting. We will start with very general (and oversimplified) categorical considerations.

Fix a ground field, k , and consider the tensor category of small k -linear DG categories, morphisms being DG functors. Fix also a *homology theory* on the latter category, i.e. a covariant tensor functor \mathbf{H} to a tensor category of modules over a commutative ring⁴ K , satisfying the following axioms:

- (1) \mathbf{H} respects quasi-equivalences.
- (2) For any DG algebra A the canonical embedding $A \rightarrow \text{Perf } A$ induces an isomorphism

$$\mathbf{H}(A) \simeq \mathbf{H}(\text{Perf } A).$$

- (3) $\mathbf{H}(k) = K$ (then, by (2), $\mathbf{H}(\text{Perf } k) = K$).

Notice that (1) and (2) together imply that \mathbf{H} descends to an invariant of noncommutative DG-schemes. Also, by the very definition of \mathbf{H} , there exists a functorial Künneth type isomorphism

$$\mathbf{H}(\mathcal{A}) \otimes_K \mathbf{H}(\mathcal{B}) \simeq \mathbf{H}(\mathcal{A} \otimes \mathcal{B}).$$

Let us add to this list one more condition:

- (4) For any DG category \mathcal{A} there is a functorial isomorphism

$${}^\vee : \mathbf{H}(\mathcal{A}) \simeq \mathbf{H}(\mathcal{A}^{\text{op}})$$

which equals identity in the case $\mathcal{A} = k$.

We will assume that the above isomorphisms satisfy all the natural properties and compatibility conditions one can imagine⁵.

To describe what we understand by an abstract HRR theorem for noncommutative DG-schemes, we need to define the Chern character map with values in the homology theory \mathbf{H} . This is a function $\text{Ch}_{\mathbf{H}}^A : \mathcal{A} \rightarrow \mathbf{H}(\mathcal{A})$, one for each DG category \mathcal{A} , defined as follows. Take an object $N \in \mathcal{A}$ and consider the DG functor $T_N : k \rightarrow \mathcal{A}$ that sends the unique object of k to N . Then [11, 36]

$$\text{Ch}_{\mathbf{H}}^A(N) = \mathbf{H}(T_N)(1_K).$$

⁴One can take \mathbb{Z} -graded, $\mathbb{Z}/2$ -graded modules, modules that are complete in some topology etc.

⁵The right definition of a homology theory should be formulated in terms of the category of noncommutative motives [40].

Clearly, the Chern character is functorial: For any two DG categories \mathcal{A}, \mathcal{B} and any DG functor $F : \mathcal{A} \rightarrow \mathcal{B}$

$$\mathrm{Ch}_{\mathbf{H}}^{\mathcal{B}} \circ F = \mathbf{H}(F) \circ \mathrm{Ch}_{\mathbf{H}}^{\mathcal{A}}.$$

From now on, we will focus on proper DG categories, i.e. DG categories that correspond to proper noncommutative DG-schemes. Let \mathcal{A} be a proper DG category. Consider the DG functor

$$\mathbf{Hom}_{\mathcal{A}} : \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}} \rightarrow \mathrm{Perf}k, \quad N \otimes M \mapsto \mathrm{Hom}_{\mathcal{A}}(M, N).$$

By (3), it induces a linear map $\mathbf{H}(\mathbf{Hom}_{\mathcal{A}}) : \mathbf{H}(\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}) \rightarrow K$. One can compose it with the Künneth isomorphism to get a K -bilinear pairing

$$\langle \cdot, \cdot \rangle_{\mathcal{A}} : \mathbf{H}(\mathcal{A}) \times \mathbf{H}(\mathcal{A}^{\mathrm{op}}) \rightarrow K.$$

Now we are ready to formulate the HRR theorem: For any proper DG category \mathcal{A} and any two objects $N, M \in \mathcal{A}$

$$(1.2) \quad \mathrm{Ch}_{\mathbf{H}}^{\mathrm{Perf}k}(\mathrm{Hom}_{\mathcal{A}}(M, N)) = \langle \mathrm{Ch}_{\mathbf{H}}^{\mathcal{A}}(N), \mathrm{Ch}_{\mathbf{H}}^{\mathcal{A}}(M)^{\vee} \rangle_{\mathcal{A}}.$$

Indeed, it follows from the functoriality of the isomorphism $^{\vee}$ that

$$(\mathbf{H}(T_M)(1_K))^{\vee} = \mathbf{H}(T_{M^{\mathrm{op}}})(1_K)$$

where M^{op} stands for M viewed as an object of $\mathcal{A}^{\mathrm{op}}$. Then

$$\begin{aligned} \langle \mathrm{Ch}_{\mathbf{H}}^{\mathcal{A}}(N), \mathrm{Ch}_{\mathbf{H}}^{\mathcal{A}}(M)^{\vee} \rangle_{\mathcal{A}} &= \mathbf{H}(\mathbf{Hom}_{\mathcal{A}}) (\mathbf{H}(T_N)(1_K) \otimes (\mathbf{H}(T_M)(1_K))^{\vee}) \\ &= \mathbf{H}(\mathbf{Hom}_{\mathcal{A}}) (\mathbf{H}(T_N)(1_K) \otimes \mathbf{H}(T_{M^{\mathrm{op}}})(1_K)) = \mathbf{H}(\mathbf{Hom}_{\mathcal{A}}) (\mathbf{H}(T_{N \otimes M^{\mathrm{op}}})(1_K)) \\ &= \mathbf{H}(\mathbf{Hom}_{\mathcal{A}} \circ T_{N \otimes M^{\mathrm{op}}})(1_K) = \mathbf{H}(\mathrm{Hom}_{\mathcal{A}}(M, N))(1_K) = \mathrm{Ch}_{\mathbf{H}}^{\mathrm{Perf}k}(\mathrm{Hom}_{\mathcal{A}}(M, N)). \end{aligned}$$

In this very general form, the HRR theorem is almost tautological. For it to be of any use, one needs to find a way to compute the right-hand side of (1.2) for a given proper noncommutative DG-scheme and any pair of perfect complexes on it. In this work, we solve this problem in the case $K = k$, $\mathbf{H} = \mathrm{HH}_{\bullet}$, where HH_{\bullet} stands for the Hochschild homology⁶ (see Section 2.3 for the definition). This choice of the homology theory can be motivated as follows.

First of all, there is a classical character map from the Grothendieck group of a ring to its Hochschild homology - the so called Dennis trace map [43]. Its sheafified version appeared in [11] in connection with the index theorem for elliptic pairs [58, 59] (the definition of the Chern character given above mimics the one given in [11]).

In the algebraic geometric context, the relevance of the Hochschild homology to the HRR theorem can be explained as follows. There is a version of the HRR theorem for compact complex manifolds [48, 49], in which the Chern class of a coherent sheaf takes values in the Hodge cohomology $\oplus_i \mathrm{H}^i(\Omega_X^i)$ (see also [27]). A new proof of this result was obtained in [44, 45] using an algebraic-differential calculus (see also [14, 50]). This latter approach emphasizes the importance of viewing the Chern character as a map to the Hochschild homology $\mathrm{HH}_0(X)$ of the space X . The “usual” Chern character is then obtained via the Hochschild-Kostant-Rosenberg isomorphism $\mathrm{HH}_0(X) \cong \oplus_i \mathrm{H}^i(\Omega_X^i)$. This

⁶The most difficult axioms (1) and (2) in our “definition” of the homology theory were proved for HH_{\bullet} by B. Keller in [34].

point of view was further developed in [13]. Namely, it was explained in [13] (see also [15]) how to obtain a categorical version of the HRR theorem, similar to the one above, starting from the cohomology theory

$$\text{smooth spaces} \rightarrow \text{graded vector spaces}, \quad X \mapsto \mathrm{HH}_\bullet(X)$$

(“smooth spaces” are understood in a broad sense: these are usual schemes as well as various almost commutative ones such as orbifolds). Finally, the transition from X to its categorical incarnation, $\mathrm{Perf} X$, is based on the fact that $\mathrm{HH}_\bullet(X)$ is isomorphic to the Hochschild homology of the DG category $\mathrm{Perf} X$, which was proved in [36].

Before we move on to the description of the main results of the paper, we would like to mention a notational convention we are going to follow.

Following [11] (see also [36]), we will call the Chern character $\mathrm{Ch}_{\mathrm{HH}}$ with values in the Hochschild homology the *Euler* character and use the notation Eu .

1.3. Main results. Let us describe the main results of this work.

Fix a ground field k and a proper DG algebra A over k (as we mentioned earlier, the properness means $\sum_n \dim H^n(A) < \infty$).

The first main result is the computation of the Euler class $\mathrm{eu}(L)$ of an arbitrary perfect DG A -module L . Here $\mathrm{eu}(L)$ stands for the unique element in $\mathrm{HH}_0(A)$ that corresponds to $\mathrm{Eu}(L) \in \mathrm{HH}_0(\mathrm{Perf} A)$ under the canonical isomorphism $\mathrm{HH}_\bullet(A) \simeq \mathrm{HH}_\bullet(\mathrm{Perf} A)$ (see axiom (2) in Section 1.2). The following theorem is proved in Section 4.1.

Theorem 1. *Let $N = (\bigoplus_j A[r_j], d + \alpha)$ be a twisted DG A -module and L a homotopy direct summand of N which corresponds to a homotopy idempotent $\pi : N \rightarrow N$. Then*

$$\mathrm{eu}(L) = \sum_{l=0}^{\infty} (-1)^l \mathrm{str}(\pi \underbrace{[\alpha | \dots | \alpha]}_l)$$

Roughly speaking, in this formula π and α are elements of a DG analog of the matrix algebra $\mathrm{Mat}(A)$, $\pi[\alpha | \dots | \alpha]$ is an element of the Hochschild chain complex of this DG matrix algebra, and str is an analog of the usual trace map $\mathrm{tr} : \mathrm{Mat}(A) \rightarrow A$ (see [24, 43]). Note that α is upper-triangular, so the series terminates.

To present our next result, we observe that the pairing

$$\mathrm{HH}_\bullet(\mathrm{Perf} A) \times \mathrm{HH}_\bullet((\mathrm{Perf} A)^{\mathrm{op}}) \rightarrow \mathrm{HH}_\bullet(\mathrm{Perf} k) \simeq k,$$

defined earlier in Section 1.2, induces a pairing

$$(1.3) \quad \mathrm{HH}_\bullet(\mathrm{Perf} A) \times \mathrm{HH}_\bullet(\mathrm{Perf} A^{\mathrm{op}}) \rightarrow k.$$

This is due to the existence of a canonical quasi-equivalence of DG categories (see (3.6)):

$$D : \mathrm{Perf} A^{\mathrm{op}} \rightarrow (\mathrm{Perf} A)^{\mathrm{op}}, \quad M \mapsto DM = \mathrm{Hom}_{\mathrm{Perf} A^{\mathrm{op}}}(M, A).$$

In fact, we “twist” the exposition in the main text (Section 3.1) and work exclusively with the pairing (1.3). The reason is that it can be defined very explicitly without referring to

its categorical nature. Besides, it induces a pairing

$$(1.4) \quad \langle , \rangle : \mathrm{HH}_\bullet(A) \times \mathrm{HH}_\bullet(A^{\mathrm{op}}) \rightarrow k$$

via the canonical isomorphisms $\mathrm{HH}_\bullet(A) \simeq \mathrm{HH}_\bullet(\mathrm{Perf} A)$, $\mathrm{HH}_\bullet(A^{\mathrm{op}}) \simeq \mathrm{HH}_\bullet(\mathrm{Perf} A^{\mathrm{op}})$. This latter pairing is described explicitly in our next theorem, which is obtained by combining results of Section 3.2 (see formulas (3.4), (3.5)) and Theorem 4.6.

Theorem 2. *Let a, b be two elements of $\mathrm{HH}_\bullet(A)$, $\mathrm{HH}_\bullet(A^{\mathrm{op}})$, respectively. Then*

$$\langle a, b \rangle = \int a \wedge b.$$

Here $\wedge : \mathrm{HH}_\bullet(A) \times \mathrm{HH}_\bullet(A^{\mathrm{op}}) \rightarrow \mathrm{HH}_\bullet(\mathrm{End}_k(A))$, $\int : \mathrm{HH}_\bullet(\mathrm{End}_k(A)) \rightarrow k$ are defined as follows:

(1) If $\sum_a a_0[a_1 | \dots | a_l]$ (resp. $\sum_b b_0[b_1 | \dots | b_m]$) is a cycle in the Hochschild chain complex of A (resp. A^{op}) representing the homology class a (resp. b) then

$$a \wedge b = \sum_{a,b} \mathrm{sh}(L(a_0)[L(a_1) | \dots | L(a_l)] \otimes R(b_0)[R(b_1) | \dots | R(b_m)]),$$

where $L(a_i)$ (resp. $R(b_j)$) stands for the operator in A of left (resp. right) multiplication with a_i (resp. b_j); sh is the well known shuffle-product (see Section 2.4).

(2) \int is what we call the Feigin-Losev-Shoikhet trace [22, 51]. It is described explicitly in Theorem 4.6 (Section 4.2).

Furthermore, recall that there should exist a canonical isomorphism ${}^\vee : \mathrm{HH}_\bullet(A) \simeq \mathrm{HH}_\bullet(A^{\mathrm{op}})$ (see axiom (4) in Section 1.2). In fact, the isomorphism is easy to describe explicitly (see Section 3.2). By summarizing the above discussion, we get the following version of the noncommutative HRR theorem:

Theorem 3. *For any perfect DG A -modules N, M*

$$\chi(M, N)(:= \chi(\mathrm{Hom}_{\mathrm{Perf} A}(M, N))) = \int \mathrm{eu}(N) \wedge \mathrm{eu}(M)^\vee.$$

The only thing that needs to be explained here is where $\chi(\mathrm{Hom}_{\mathrm{Perf} A}(M, N))$ came from. According the categorical HRR theorem, described in the previous section, the left-hand of the above equality should equal $\mathrm{eu}(\mathrm{Hom}_{\mathrm{Perf} A}(M, N))$. However, the Euler class of a perfect DG k -module is nothing but its Euler characteristic. This is a consequence of the following “expected” fact, which we prove in Section 3.1: for any A the Euler character Eu descends to a character on the Grothendieck group of the triangulated category $\mathrm{Ho}(\mathrm{Perf} A)$, the homotopy category of $\mathrm{Perf} A$.

Note that the noncommutative HRR formula doesn’t include any analog of the Todd class. The Todd class seems to emerge in the case when a noncommutative space, \widehat{X} , is “close” to a commutative one, X (for example, \widehat{X} is a deformation quantization of X). In such cases various homology theories of \widehat{X} can be identified with certain cohomology

rings associated with X and the Todd class of X appears because of this identification. For some classes of noncommutative spaces one can try to define an analog of the Todd class “by hand” but, in general, a categorical definition of the Todd class doesn’t seem to exist.

In the main text we do not refer to the categorical version of the HRR theorem to prove Theorem 3. Instead, we derive it from a more general statement (Theorem 3.4). Roughly speaking, this statement says the following: If A and B are two proper DG algebras and X is a perfect $A - B$ -bimodule then the map $\mathrm{HH}_\bullet(\mathrm{Perf} A) \rightarrow \mathrm{HH}_\bullet(\mathrm{Perf} B)$, induced by the DG functor $- \otimes_A X : \mathrm{Perf} A \rightarrow \mathrm{Perf} B$, is given by a “convolution” with the Euler class of X . Later, in Section 6.1, we use this result again to prove the following

Theorem 4. *Let A be a proper smooth DG algebra. Then the pairing \langle , \rangle is non-degenerate.*

It is this application that was the original motivation for the author to study the Euler classes in the DG setting ⁷. It implies, in particular, the noncommutative Hodge-to-De Rham degeneration conjecture [41] for smooth algebras with the trivial differential and grading. Hopefully, the reader will accept all this as an excuse for “twisting” the exposition and not mentioning the categorical HRR in what follows.

In the end of this work we present some “toy” examples of proper noncommutative DG-schemes and the HRR formulas for them. Namely, in Section 5.1 we discuss what we call directed algebras. Basically, these are some quiver algebras with relations but we find the quiver-free description more convenient when it comes to proving general facts about such algebras. Many commutative schemes, viewed as noncommutative ones, are described by directed algebras. Namely, this is so when the scheme possesses a strongly exceptional collection [6]. The HRR formula for such algebras (see (5.3)) is a special case of Ringel’s formula [54, Section 2.4]. Section 5.2 is about proper noncommutative DG-schemes “responsible” for orbifold singularities of the form \mathbb{C}^n/G , where G is a finite subgroup of $SL_n(\mathbb{C})$. Namely, we look at the noncommutative DG-scheme related to the derived category of complexes of G -equivariant coherent sheaves on \mathbb{C}^n with supports at the origin. We conjecture that the underlying DG algebra is the cross-product $\Lambda^\bullet \mathbb{C}^n \rtimes \mathbb{C}[G]$ and we derive the HRR formula for some perfect modules over this algebra (see (5.4)).

Section 6.2 is devoted to a less straightforward application of our results. It has been conjectured by Y. Soibelman and K. Costello that for a Calabi-Yau DG algebra the pairing $\mathrm{HH}_\bullet(A) \times \mathrm{HH}_\bullet(A^{\mathrm{op}}) \rightarrow k$, we construct in this paper, coincides with the one coming from the Topological Field Theory associated with A by [18, 41]. In Section 6.2 we formulate this conjecture and verify it in the particular case of Frobenius algebras.

1.4. Other viewpoints on the noncommutative HRR theorem. In this section, we provide a very brief account of other Riemann-Roch type theorems in Noncommutative Geometry we are aware of.

⁷I am grateful to Y. Soibelman for suggesting to me to “write up” the proof of this statement.

Let us begin with the afore-mentioned preprint [13] which partially inspired the present work. The approach taken in [13] is based on an alternative description of the Hochschild homology of a *smooth* proper space X in terms of the Serre functor $S_X : D^b(X) \rightarrow D^b(X)$. Namely,

$$\mathrm{HH}_\bullet(X) \simeq \mathrm{Ext}_{\mathrm{Fun}}^\bullet(S_X^{-1}, I_X),$$

where I_X is the identity endofunctor of $D^b(X)$ and the extensions are taken in a suitably defined triangulated category of endofunctors. In [60] we generalized the above isomorphism to the case of an arbitrary smooth proper noncommutative DG-scheme. However, proving that the above definition gives rise to a homology theory on the category of smooth proper noncommutative DG-schemes (in other words, lifting the above definition on the level of DG categories) will require some efforts [13, Appendix B]. Besides, the “traditional” definition of the Hochschild homology we use in this paper works for an arbitrary, not necessarily smooth scheme.

Other analogs of the Riemann-Roch theorem were obtained in [28], [46] in the framework of Noncommutative Algebraic Geometry [1], [55], [56], [69]. The exposition in [46] is closer to ours in that it emphasizes the importance of triangulated categories in connection with Riemann-Roch type results. Our approach and the above two approaches to the noncommutative Riemann-Roch theorem are not completely unrelated since many interesting noncommutative schemes give rise to noncommutative DG schemes [10, Section 4].

Last, but not least, various index theorems have been proved in frameworks of A. Connes’ Noncommutative Geometry [17, 61] and Deformation Quantization [11, 21, 47, 67].

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2. HOCHSCHILD HOMOLOGY OF DG ALGEBRAS AND DG CATEGORIES

2.1. DG algebras, DG categories, and DG functors. Throughout the paper, we work over a fixed ground field k . All vector spaces, algebras, linear categories are defined over k .

We consider unital DG algebras with no restrictions on the \mathbb{Z} -grading. If A is a DG algebra

$$A = \bigoplus_{n \in \mathbb{Z}} A^n, \quad d = d_A : A^n \rightarrow A^{n+1}$$

then A^{op} will stand for the opposite DG algebra, i.e. A^{op} coincides with A as a complex of vector spaces and the product on A^{op} is given by

$$a' \otimes a'' \mapsto (-1)^{|a'| |a''|} a'' a'$$

(here and further, $|n|$ denotes the degree of a homogeneous element n of a graded vector space). $\mathbf{Mod}A$ will stand for the DG category of right DG A -modules.

The homotopy category of a DG category \mathcal{A} will be denoted by $\mathbf{Ho}(\mathcal{A})$. Let us recall the definition of the standard triangulated structure on $\mathbf{Ho}(\mathbf{Mod}A)$. The shift functor is defined in the obvious way. The distinguished triangles are defined as follows. Let $p : L \rightarrow M$ be a degree 0 closed morphism. The cone $\mathbf{Cone}(p)$ of the morphism p is a DG A -module defined by

$$\mathbf{Cone}(p) = \left(\begin{array}{c} L[1] \\ \oplus \\ M \end{array}, \begin{pmatrix} d_{L[1]} & 0 \\ p & d_M \end{pmatrix} \right)$$

(the direct sum is taken in the category of *graded* A -modules). There are obvious degree 0 closed morphisms $q : M \rightarrow \mathbf{Cone}(p)$ and $r : \mathbf{Cone}(p) \rightarrow L[1]$. A triangle in $\mathbf{Ho}(\mathbf{Mod}A)$ is, by definition, a sequence $X \rightarrow Y \rightarrow Z \rightarrow X[1]$ isomorphic to a sequence of the form $L \xrightarrow{p} M \xrightarrow{q} \mathbf{Cone}(p) \xrightarrow{r} L[1]$.

Let N be a right DG A -module. A degree 0 closed endomorphism $\pi : N \rightarrow N$ will be called a homotopy idempotent if $\pi^2 = \pi$ in $\mathbf{Ho}(\mathbf{Mod}A)$. By a homotopy direct summand of N we will understand a DG A -module L that satisfies the following property: there exists a homotopy idempotent $\pi : N \rightarrow N$ and two degree 0 morphisms $f : N \rightarrow L$ and $g : L \rightarrow N$ such that $fg = 1_L$, $gf = \pi$ in $\mathbf{Ho}(\mathbf{Mod}A)$.

Fix two DG categories \mathcal{A} and \mathcal{B} and consider the DG category $\mathbf{Fun}(\mathcal{A}, \mathcal{B})$ of DG functors from \mathcal{A} to \mathcal{B} [35]. A degree 0 closed morphism $f \in \mathbf{Hom}_{\mathbf{Fun}(\mathcal{A}, \mathcal{B})}(F, G)$ will be called a *weak homotopy equivalence* if for any $N \in \mathcal{A}$ the morphism $f(N) : F(N) \rightarrow G(N)$ is a homotopy equivalence in $\mathbf{Ho}(\mathcal{B})$.

2.2. Perfect modules. Let A be a DG algebra. It can be viewed as a full DG subcategory of $\mathbf{Mod}A$ with a single object. The embedding $A \hookrightarrow \mathbf{Mod}A$ factors through a smaller full subcategory $\mathbf{Perf}A \subset \mathbf{Mod}A$ of perfect A -modules. This subcategory is defined as follows (see [8]).

Let us say that a DG A -module N is *finitely generated free* if it is isomorphic to a module of the form $K \otimes A$ where K is a finite dimensional graded vector space (equivalently, it is a finite direct sum of shifts of A). We will say that $N \in \mathbf{Mod}A$ is *finitely generated semi-free* if it can be obtained from a finite set of finitely generated free A -modules (equivalently, a finite set of shifts of A) by successive taking the cones of degree 0 closed morphisms. Finally, a *perfect* DG A -module is a homotopy direct summand of a finitely generated semi-free DG A -module.

Note that this definition is slightly more general than the one given in [8]. The authors of [8] require perfect modules to be semi-free but we don't. For example, a complex of vector spaces is perfect in our sense iff it has finite dimensional total cohomology and it is perfect in the sense of [8] if, in addition, it is bounded above. The reason we prefer not to

restrict ourselves to semi-free modules will be clear from Proposition 2.4 below. It suffices for us to stay within the class of homotopically projective modules: N is homotopically projective iff $\mathrm{Hom}_{\mathrm{Mod} A}(N, L)$ is acyclic whenever L is acyclic. Every finitely generated semi-free module N is known to be homotopically projective [20, Section 13]. It follows that every perfect module in our sense is homotopically projective as well.

The following result is well known (and is not hard to prove):

Proposition 2.1. *The DG category $\mathrm{Perf} A$ is closed under passing to homotopically equivalent modules, taking shifts and cones of degree 0 morphisms, and taking homotopy direct summands.*

Let us list some simple useful facts about DG functors between the categories of perfect modules.

Proposition 2.2. *Let A, B be DG algebras and $F : \mathrm{Mod} A \rightarrow \mathrm{Mod} B$ a DG functor. The DG functor F preserves the subcategories of perfect modules iff $F(A) \in \mathrm{Perf} B$.*

To prove this proposition, observe that F preserves homotopy direct summands and cones of degree 0 morphisms.

For two DG algebras A, B and a bimodule $X \in \mathrm{Mod}(A^{\mathrm{op}} \otimes B)$ let us denote by T_X the DG functor

$$- \otimes_A X : \mathrm{Mod} A \rightarrow \mathrm{Mod} B.$$

Here is a straightforward consequence of the last proposition:

Corollary 2.3. *Suppose a bimodule $X \in \mathrm{Mod}(A^{\mathrm{op}} \otimes B)$ is perfect as a DG B -module. Then T_X preserves perfect modules.*

Recall that a DG algebra A is called *proper* if $\sum_n \dim H^n(A) < \infty$.

Proposition 2.4. *Let A be a proper DG algebra and B an arbitrary DG algebra. Then for any $X \in \mathrm{Perf}(A^{\mathrm{op}} \otimes B)$ the DG functor T_X preserves perfect modules.*

In view of the above corollary, it is enough to show that X is perfect as a DG B -module. Suppose that X is a homotopy direct summand of a finitely generated semi-free DG $A^{\mathrm{op}} \otimes B$ -module Y and Y is obtained from $(A^{\mathrm{op}} \otimes B)[m_1], \dots, (A^{\mathrm{op}} \otimes B)[m_l]$ by successive taking cones of degree 0 closed morphisms. As a B -module, $A^{\mathrm{op}} \otimes B$ is homotopically equivalent to the finitely generated free module $H^\bullet(A) \otimes B$ (this is where we use the properness of A and the fact that we are working over a field!). Thus, as a B -module, Y is homotopically equivalent to a finitely generated semi-free module. Then X , as a B -module, is a homotopy direct summand of a module that is homotopy equivalent to a finitely generated semi-free module. This, together with Proposition 2.1, finishes the proof.

Let us recall one more result about perfect modules which we will need later on. The fact that perfect modules are homotopically projective implies the following result (cf. [4, Corollary 10.12.4.4]):

Proposition 2.5. *If P is a perfect right DG A -module then $P \otimes_A N$ is acyclic for every acyclic DG A^{op} -module N .*

2.3. Hochschild homology. We begin by recalling the definition of the Hochschild homology groups $\text{HH}_n(A)$, $n \in \mathbb{Z}$, of a DG algebra A .

Let us use the notation sa to denote an element $a \in A$ viewed as an element of the “suspension” $sA = A[1]$. Thus, $|sa| = |a| - 1$. Let $\mathbf{C}_\bullet(A) = A \otimes T(A[1]) = \bigoplus_{n=0}^{\infty} A \otimes A[1]^{\otimes n}$ equipped with the induced grading. We will denote elements of $A \otimes A[1]^{\otimes n}$ by a_0 , if $n = 0$, and $a_0[a_1|a_2|\dots|a_n]$ otherwise (i.e. $a_0[a_1|a_2|\dots|a_n] = a_0 \otimes sa_1 \otimes sa_2 \otimes \dots \otimes sa_n$). $\mathbf{C}_\bullet(A)$ is equipped with the differential $b = b_0 + b_1$, where b_0 and b_1 are two anti-commuting differentials given by

$$(2.1) \quad b_0(a_0) = da_0, \quad b_1(a_0) = 0,$$

and

$$\begin{aligned} b_0(a_0[a_1|a_2|\dots|a_n]) &= da_0[a_1|a_2|\dots|a_n] - \sum_{i=1}^n (-1)^{\eta_{i-1}} a_0[a_1|a_2|\dots|da_i|\dots|a_n], \\ b_1(a_0[a_1|a_2|\dots|a_n]) &= (-1)^{|a_0|} a_0 a_1[a_2|\dots|a_n] + \sum_{i=1}^{n-1} (-1)^{\eta_i} a_0[a_1|a_2|\dots|a_i a_{i+1}|\dots|a_n] \\ &\quad - (-1)^{\eta_{n-1}(|a_n|+1)} a_n a_0[a_1|a_2|\dots|a_{n-1}] \end{aligned}$$

for $n \neq 0$. Here $\eta_i = |a_0| + |sa_1| + \dots + |sa_i|$. $\mathbf{C}_\bullet(A)$ is called the Hochschild chain complex of A . Then

$$\text{HH}_n(A) = H^n(\mathbf{C}_\bullet(A)).$$

Let \mathcal{A} be a (small) DG category. Its Hochschild chain complex is defined as follows. Fix a non-negative integer n . We will denote the set of sequences $\{X_0, X_1, \dots, X_n\}$ of objects of \mathcal{A} by \mathcal{A}^{n+1} (the objects in the sequence are not required to be different). Fix an element $\mathbb{X} = \{X_0, X_1, \dots, X_n\} \in \mathcal{A}^{n+1}$ and denote by $\mathbf{C}_\bullet(\mathcal{A}, \mathbb{X})$ the graded vector space $\text{Hom}_{\mathcal{A}}(X_n, X_0) \otimes \text{Hom}_{\mathcal{A}}(X_{n-1}, X_n)[1] \otimes \dots \otimes \text{Hom}_{\mathcal{A}}(X_0, X_1)[1]$. Equip the space

$$\mathbf{C}_\bullet(\mathcal{A}) = \bigoplus_{n \geq 0} \bigoplus_{\mathbb{X} \in \mathcal{A}^{n+1}} \mathbf{C}_\bullet(\mathcal{A}, \mathbb{X})$$

with the differential $b = b_0 + b_1$ where b_0 and b_1 are given by formulas analogous to (2.1),(2.2). The complex $\mathbf{C}_\bullet(\mathcal{A})$ is the Hochschild chain complex of the DG category \mathcal{A} and its cohomology

$$\text{HH}_n(\mathcal{A}) = H^n(\mathbf{C}_\bullet(\mathcal{A}))$$

is the Hochschild homology of \mathcal{A} .

Obviously, any DG functor $F : \mathcal{A} \rightarrow \mathcal{B}$ between two DG categories \mathcal{A}, \mathcal{B} induces a morphism of complexes $\mathbf{C}(F) : \mathbf{C}_\bullet(\mathcal{A}) \rightarrow \mathbf{C}_\bullet(\mathcal{B})$ and, as a result, a linear map

$$\text{HH}(F) : \text{HH}_\bullet(\mathcal{A}) \rightarrow \text{HH}_\bullet(\mathcal{B}).$$

Being applied to the embedding $A \rightarrow \text{Perf } A$, the above construction yields a morphism of complexes $\mathbf{C}_\bullet(A) \rightarrow \mathbf{C}_\bullet(\text{Perf } A)$. The following result was proved in [34] (see also [35]):

Theorem 2.6. *The morphism $C_\bullet(A) \rightarrow C_\bullet(\text{Perf } A)$ is a quasi-isomorphism.*

Later on, we will need yet another result proved in [34] (see Section 3.4 of loc.cit.):

Theorem 2.7. *Let A and B be two DG algebras and $F, G : \text{Perf } A \rightarrow \text{Perf } B$ two DG functors. If there is a weak homotopy equivalence $F \rightarrow G$ then $\text{HH}(F) = \text{HH}(G)$.*

2.4. Künneth isomorphism. Let us recall the construction of the Künneth isomorphism

$$\bigoplus_n \text{HH}_n(A) \otimes \text{HH}_{N-n}(B) \simeq \text{HH}_N(A \otimes B)$$

where A, B are two DG algebras. The formula below is borrowed from [43] (see also [67] where the differential graded case is discussed).

Let us fix a DG algebra A . The first ingredient of the construction is the shuffle product

$$\text{sh} : C_\bullet(A) \otimes C_\bullet(A) \rightarrow C_\bullet(A)$$

defined as follows. For two elements $a'_0[a'_1|a'_2|\dots|a'_n], a''_0[a''_1|a''_2|\dots|a''_m] \in C_\bullet(A)$ the shuffle product is given by the formula:

$$(2.2) \quad \text{sh}(a'_0[a'_1|a'_2|\dots|a'_n] \otimes a''_0[a''_1|a''_2|\dots|a''_m]) = (-1)^* \cdot a'_0 a''_0 \text{sh}_{nm}[a'_1|\dots|a'_n|a''_1|\dots|a''_m]$$

Here $*$ is $|a''_0|(|sa'_1| + \dots + |sa'_n|)$ and

$$\text{sh}_{nm}[x_1|\dots|x_n|x_{n+1}|\dots|x_{n+m}] = \sum_{\sigma} \pm [x_{\sigma^{-1}(1)}|\dots|x_{\sigma^{-1}(n)}|x_{\sigma^{-1}(n+1)}|\dots|x_{\sigma^{-1}(n+m)}]$$

where the sum is taken over all permutations that don't shuffle the first n and the last m elements and the sign in front of each summand is computed according to the following rule: for two homogeneous elements x, y , the transposition $[\dots|x|y|\dots] \rightarrow [\dots|y|x|\dots]$ contributes $(-1)^{|x||y|}$ to the sign.

Now let B be another DG algebra. Denote by ι^A, ι^B the natural embeddings

$$A \rightarrow A \otimes B, \quad B \rightarrow A \otimes B.$$

They induce morphisms of complexes:

$$C(\iota^A) : C_\bullet(A) \rightarrow C_\bullet(A \otimes B), \quad C(\iota^B) : C_\bullet(B) \rightarrow C_\bullet(A \otimes B).$$

Theorem 2.8. *The composition K of the maps*

$$C_\bullet(A) \otimes C_\bullet(B) \xrightarrow{C(\iota^A) \otimes C(\iota^B)} C_\bullet(A \otimes B) \otimes C_\bullet(A \otimes B) \xrightarrow{\text{sh}} C_\bullet(A \otimes B)$$

respects the differentials and induces a quasi-isomorphism of complexes.

The morphism $K : C_\bullet(A) \otimes C_\bullet(B) \rightarrow C_\bullet(A \otimes B)$ defined above admits a generalization to the case of DG categories. Namely, let \mathcal{A} and \mathcal{B} be two (small) DG categories. Fix a set $\{X_0, X_1, \dots, X_n\}$ of objects of \mathcal{A} and a set $\{Y_0, Y_1, \dots, Y_m\}$ of objects of \mathcal{B} . For two elements

$$\begin{aligned} f_n[f_{n-1}|\dots|f_0] &\in \text{Hom}_{\mathcal{A}}(X_n, X_0) \otimes \text{Hom}_{\mathcal{A}}(X_{n-1}, X_n)[1] \otimes \dots \otimes \text{Hom}_{\mathcal{A}}(X_0, X_1)[1], \\ g_m[g_{m-1}|\dots|g_0] &\in \text{Hom}_{\mathcal{B}}(Y_m, Y_0) \otimes \text{Hom}_{\mathcal{B}}(Y_{m-1}, Y_m)[1] \otimes \dots \otimes \text{Hom}_{\mathcal{B}}(Y_0, Y_1)[1] \end{aligned}$$

define $K(f_n[f_{n-1}|f_{n-2}|\dots|f_0] \otimes g_m[g_{m-1}|g_{m-2}|\dots|g_0])$ as

$$\pm(f_n \otimes g_m) \text{sh}_{nm}[f_{n-1}|\dots|f_0|g_{m-1}|\dots|g_0],$$

where the sign is computed as before and sh_{nm} is defined by the formula

$$\begin{aligned} & [f_{n-1} \otimes 1_{Y_m}|\dots|f_0 \otimes 1_{Y_m}|1_{X_0} \otimes g_{m-1}|\dots|1_{X_0} \otimes g_0] + \\ & + (-1)^{|sf_0||sg_{m-1}|}[f_{n-1} \otimes 1_{Y_m}|\dots|1_{X_1} \otimes g_{m-1}|f_0 \otimes 1_{Y_{m-1}}|\dots|1_{X_0} \otimes g_0] + \dots \end{aligned}$$

Other terms in this sum are obtained from the first one by shuffling the f -terms with the g -terms according to the following rule:

$$[\dots|f_k \otimes 1_{Y_{l+1}}|1_{X_k} \otimes g_l|\dots] \rightarrow (-1)^{|sf_k||sg_l|}[\dots|1_{X_{k+1}} \otimes g_l|f_k \otimes 1_{Y_l}|\dots]$$

Let A and B be two DG algebras. We have the obvious embedding of DG categories

$$\text{Perf } A \otimes \text{Perf } B \rightarrow \text{Perf}(A \otimes B),$$

which induces a morphism of complexes

$$C_\bullet(\text{Perf } A \otimes \text{Perf } B) \rightarrow C_\bullet(\text{Perf}(A \otimes B)).$$

Let us denote the composition

$$(2.3) \quad C_\bullet(\text{Perf } A) \otimes C_\bullet(\text{Perf } B) \xrightarrow{K} C_\bullet(\text{Perf } A \otimes \text{Perf } B) \rightarrow C_\bullet(\text{Perf}(A \otimes B))$$

by the same letter K . As an immediate corollary of Theorems 2.6 and 2.8, we get the following result:

Proposition 2.9. *The map $K : C_\bullet(\text{Perf } A) \otimes C_\bullet(\text{Perf } B) \rightarrow C_\bullet(\text{Perf}(A \otimes B))$ is a quasi-isomorphism.*

Indeed, we have the commutative diagram

$$\begin{array}{ccc} C_\bullet(\text{Perf } A) \otimes C_\bullet(\text{Perf } B) & \longrightarrow & C_\bullet(\text{Perf}(A \otimes B)) \\ \uparrow & & \uparrow \\ C_\bullet(A) \otimes C_\bullet(B) & \longrightarrow & C_\bullet(A \otimes B) \end{array}$$

in which the vertical arrows and the arrow on the bottom are quasi-isomorphisms.

Finally, we will formulate two more results about the Künneth map (2.3). Both results follow directly from the definition of K .

Proposition 2.10. *Let A , B , and C be three DG algebras. The diagram*

$$\begin{array}{ccc} C_\bullet(\text{Perf } A) \otimes C_\bullet(\text{Perf } B) \otimes C_\bullet(\text{Perf } C) & \xrightarrow{K \otimes 1} & C_\bullet(\text{Perf}(A \otimes B)) \otimes C_\bullet(\text{Perf } C) \\ \downarrow 1 \otimes K & & \downarrow K \\ C_\bullet(\text{Perf } A) \otimes C_\bullet(\text{Perf}(B \otimes C)) & \xrightarrow{K} & C_\bullet(\text{Perf}(A \otimes B \otimes C)) \end{array}$$

commutes. In other words, K is associative.

Proposition 2.11. *Let A, B, C, D be DG algebras. Let $X \in \text{Mod}(A^{\text{op}} \otimes C)$ and $Y \in \text{Mod}(B^{\text{op}} \otimes D)$ be bimodules satisfying the conditions of Corollary 2.3. Then the diagram*

$$\begin{array}{ccc} \mathbf{C}_\bullet(\text{Perf } A) \otimes \mathbf{C}_\bullet(\text{Perf } B) & \xrightarrow{\mathbf{K}} & \mathbf{C}_\bullet(\text{Perf}(A \otimes B)) \\ \downarrow \mathbf{C}(T_X) \otimes \mathbf{C}(T_Y) & & \downarrow \mathbf{C}(T_{X \otimes_k Y}) \\ \mathbf{C}_\bullet(\text{Perf } C) \otimes \mathbf{C}_\bullet(\text{Perf } D) & \xrightarrow{\mathbf{K}} & \mathbf{C}_\bullet(\text{Perf}(C \otimes D)) \end{array}$$

commutes.

3. HIRZEBRUCH-RIEMANN-ROCH THEOREM

3.1. Euler character. Let A be a DG algebra and N a perfect right DG A -module. Consider the DG functor $T_N = - \otimes_k N : \text{Perf } k \rightarrow \text{Perf } A$. The Euler class $\text{Eu}(N) \in \text{HH}_0(\text{Perf } A)$ is defined by the formula (cf. [11],[34])

$$\text{Eu}(N) = \text{HH}(T_N)(1).$$

In other words, $\text{Eu}(N)$ is the class of the identity morphism 1_N in $\text{HH}_0(\text{Perf } A)$.

Let us list some basic properties of the Euler character map.

The following statement follows from Theorem 2.7:

Proposition 3.1. *If $N, M \in \text{Perf } A$ are homotopically equivalent then $\text{Eu}(N) = \text{Eu}(M)$. In other words, Eu descends to objects of $\text{Ho}(\text{Perf } A)$.*

The following result means that the Euler class descends to the Grothendieck group of the triangulated category $\text{Ho}(\text{Perf } A)$.

Proposition 3.2. *For any $N \in \text{Perf } A$ one has $\text{Eu}(N[1]) = -\text{Eu}(N)$ and for any triangle $L \xrightarrow{p} M \xrightarrow{q} N \xrightarrow{r} L[1]$ in $\text{Ho}(\text{Perf } A)$ one has*

$$(3.1) \quad \text{Eu}(M) = \text{Eu}(L) + \text{Eu}(N).$$

Let us prove the first part. We have to show that $1_N + 1_{N[1]}$ is homologous to 0 in $\mathbf{C}_\bullet(\text{Perf } A)$. Denote by $1_{N, N[1]}$ (resp. $1_{N[1], N}$) the identity endomorphism of N viewed as a morphism from N to $N[1]$ (resp. from $N[1]$ to N). Then

$$\begin{aligned} b(1_{N, N[1]}[1_{N[1], N}]) &= b_1(1_{N, N[1]}[1_{N[1], N}]) = \\ &= -(1_{N, N[1]}1_{N[1], N} + 1_{N[1], N}1_{N, N[1]}) = -(1_{N[1]} + 1_N) \end{aligned}$$

Let us prove the second part. By Proposition 3.1, it suffices to prove (3.1) for $N = \text{Cone}(p)$. Consider the following morphisms:

$$\begin{aligned} j_1 &= \begin{pmatrix} 1_{L[1]} \\ 0 \end{pmatrix} : L[1] \rightarrow \text{Cone}(p), & q_1 &= \begin{pmatrix} 1_{L[1]} & 0 \end{pmatrix} : \text{Cone}(p) \rightarrow L[1], \\ j_2 &= \begin{pmatrix} 0 \\ 1_M \end{pmatrix} : M \rightarrow \text{Cone}(p), & q_2 &= \begin{pmatrix} 0 & 1_M \end{pmatrix} : \text{Cone}(p) \rightarrow M. \end{aligned}$$

It is easy to see that

$$d(j_1) = j_2 \cdot p, \quad d(q_1) = 0, \quad d(j_2) = 0, \quad d(q_2) = -p \cdot q_1.$$

(In these formulas, p is viewed as a degree 1 morphism from $L[1]$ to M .) The following computation finishes the proof:

$$\begin{aligned} 1_{\text{Cone}(p)} - 1_{L[1]} - 1_M &= j_1 q_1 + j_2 q_2 - q_1 j_1 - q_2 j_2 = [j_1, q_1] + [j_2, q_2] \\ &= b(j_1[q_1] + j_2[q_2]) - b_0(j_1[q_1] + j_2[q_2]) = b(j_1[q_1] + j_2[q_2]) - (d(j_1)[q_1] - j_2[d(q_2)]) \\ &= b(j_1[q_1] + j_2[q_2]) - (j_2 p[q_1] + j_2[p q_1]) = b(j_1[q_1] + j_2[q_2] - j_2[p q_1]). \end{aligned}$$

To formulate the main result of this section, we need a pairing

$$\text{HH}_n(\text{Perf } A) \times \text{HH}_{-n}(\text{Perf } A^{\text{op}}) \rightarrow k, \quad n \in \mathbb{Z},$$

where A is a proper DG algebra. Here is the definition.

Let us equip A with a left DG $A \otimes A^{\text{op}}$ -module structure as follows:

$$(a' \otimes a'')a = (-1)^{|a''||a|} a' a a''.$$

We will denote the resulting A -bimodule by Δ .

Consider the DG functor:

$$T_\Delta : \text{Mod}(A \otimes A^{\text{op}}) \rightarrow \text{Mod } k, \quad N \mapsto N \otimes_{A \otimes A^{\text{op}}} A$$

The following proposition is an immediate consequence of Corollary 2.3.

Proposition 3.3. *If A is proper then T_Δ induces a DG functor $\text{Perf}(A \otimes A^{\text{op}}) \rightarrow \text{Perf } k$.*

We can use this to define a pairing

$$(3.2) \quad \langle \cdot, \cdot \rangle : \text{HH}_n(\text{Perf } A) \times \text{HH}_{-n}(\text{Perf } A^{\text{op}}) \rightarrow k, \quad n \in \mathbb{Z}$$

via the composition of morphisms of complexes

$$\text{C}_\bullet(\text{Perf } A) \otimes \text{C}_\bullet(\text{Perf } A^{\text{op}}) \xrightarrow{\text{K}} \text{C}_\bullet(\text{Perf}(A \otimes A^{\text{op}})) \xrightarrow{\text{C}(T_\Delta)} \text{C}_\bullet(\text{Perf } k)$$

and the fact that $\text{HH}_n(\text{Perf } k) \simeq \text{HH}_n(k)$ is k , if $n = 0$, and 0 otherwise.

Before we formulate the main result of this section, let us introduce the following notation. For a bimodule $X \in \text{Perf}(A^{\text{op}} \otimes B)$ we will denote by $\text{Eu}'(X)$ the element

$$\text{K}^{-1}(\text{Eu}(X)) \in \bigoplus_n \text{HH}_{-n}(\text{Perf } A^{\text{op}}) \otimes \text{HH}_n(\text{Perf } B),$$

where K is the Künneth isomorphism.

Theorem 3.4. *Let A be a proper DG algebra, B an arbitrary DG algebra, and X any object of $\text{Perf}(A^{\text{op}} \otimes B)$. If $y \in \text{HH}_\bullet(\text{Perf } A)$ then $\text{HH}(T_X)(y) = \langle y, \text{Eu}'(X) \rangle$. That is, if*

$$\text{Eu}'(X) = \sum_n x'_{-n} \otimes x''_n \in \bigoplus_n \text{HH}_{-n}(\text{Perf } A^{\text{op}}) \otimes \text{HH}_n(\text{Perf } B),$$

then $\text{HH}(T_X)(y) = \sum_n \langle y, x'_{-n} \rangle \cdot x''_n$.

To prove this, observe that T_X can be described as a composition of the following DG functors

$$\mathrm{Perf} A \xrightarrow{-\otimes_k X} \mathrm{Perf}(A \otimes A^{\mathrm{op}} \otimes B) \xrightarrow{T_{\Delta \otimes_k B}} \mathrm{Perf} B$$

Thus, $\mathrm{HH}(T_X) = \mathrm{HH}(T_{\Delta \otimes_k B}) \circ \mathrm{HH}(-\otimes_k X)$. It follows from the definition of the Künneth isomorphism K that the diagram

$$\begin{array}{ccc} \mathrm{HH}_\bullet(\mathrm{Perf} A) & \xrightarrow{\mathrm{HH}(-\otimes_k X)} & \mathrm{HH}_\bullet(\mathrm{Perf}(A \otimes A^{\mathrm{op}} \otimes B)) \\ 1 \otimes \mathrm{Eu}(X) \downarrow & \nearrow K & \\ \mathrm{HH}_\bullet(\mathrm{Perf} A) \otimes \mathrm{HH}_0(\mathrm{Perf}(A^{\mathrm{op}} \otimes B)) & & \end{array}$$

commutes. By conjugating with $1 \otimes K$, we get the following commutative diagram:

$$\begin{array}{ccc} \mathrm{HH}_\bullet(\mathrm{Perf} A) & \xrightarrow{\mathrm{HH}(-\otimes_k X)} & \mathrm{HH}_\bullet(\mathrm{Perf}(A \otimes A^{\mathrm{op}} \otimes B)) \\ 1 \otimes \mathrm{Eu}'(X) \downarrow & \nearrow K \circ (1 \otimes K) & \\ \mathrm{HH}_\bullet(\mathrm{Perf} A) \otimes \mathrm{HH}_\bullet(\mathrm{Perf} A^{\mathrm{op}}) \otimes \mathrm{HH}_\bullet(\mathrm{Perf} B) & & \end{array}$$

Furthermore, by Proposition 2.11 the diagram

$$\begin{array}{ccc} \mathrm{HH}_\bullet(\mathrm{Perf}(A \otimes A^{\mathrm{op}} \otimes B)) & \xrightarrow{\mathrm{HH}(T_{\Delta \otimes_k B})} & \mathrm{HH}_\bullet(\mathrm{Perf}(k \otimes B)) \simeq \mathrm{HH}_\bullet(\mathrm{Perf} B) \\ K^{-1} \downarrow & & \uparrow K \\ \mathrm{HH}_\bullet(\mathrm{Perf}(A \otimes A^{\mathrm{op}})) \otimes \mathrm{HH}_\bullet(\mathrm{Perf} B) & \xrightarrow{\mathrm{HH}(T_\Delta) \otimes 1} & \mathrm{HH}_\bullet(\mathrm{Perf} k) \otimes \mathrm{HH}_\bullet(\mathrm{Perf} B) \end{array}$$

commutes. Conjugating with $K \otimes 1$ gives us the following commutative diagram:

$$\begin{array}{ccc} \mathrm{HH}_\bullet(\mathrm{Perf}(A \otimes A^{\mathrm{op}} \otimes B)) & \xrightarrow{\mathrm{HH}(T_{\Delta \otimes_k B})} & \mathrm{HH}_\bullet(\mathrm{Perf}(k \otimes B)) \simeq \mathrm{HH}_\bullet(\mathrm{Perf} B) \\ (K^{-1} \otimes 1) K^{-1} \downarrow & & \uparrow K \\ \mathrm{HH}_\bullet(\mathrm{Perf} A) \otimes \mathrm{HH}_\bullet(\mathrm{Perf} A^{\mathrm{op}}) \otimes \mathrm{HH}_\bullet(\mathrm{Perf} B) & \xrightarrow{(\mathrm{HH}(T_\Delta) K) \otimes 1} & \mathrm{HH}_\bullet(\mathrm{Perf} k) \otimes \mathrm{HH}_\bullet(\mathrm{Perf} B) \end{array}$$

By concatenating the top arrows of the former and the latter diagrams, we get the following result:

$$\mathrm{HH}(T_{\Delta \otimes_k B}) \circ \mathrm{HH}(-\otimes_k X) = K \circ ((\mathrm{HH}(T_\Delta) K) \otimes 1) \circ (K^{-1} \otimes 1) \circ K^{-1} \circ K \circ (1 \otimes K) \circ (1 \otimes \mathrm{Eu}'(X)).$$

By associativity of the Künneth isomorphism (Proposition 2.10), the latter product is nothing but $K \circ ((\mathrm{HH}(T_\Delta) K) \otimes 1) \circ (1 \otimes \mathrm{Eu}'(X))$ which finishes the proof.

Theorem 3.4 generates several corollaries. The first one, the Hirzebruch-Riemann-Roch type formula, will be formulated and proved in the next section. Another corollary, which concerns the so-called smooth DG algebras, will be described in Section 6.1.

3.2. Hirzebruch-Riemann-Roch theorem. Essentially, the Hirzebruch-Riemann-Roch theorem is the following result:

Theorem 3.5. *Let A be a proper DG algebra. Then, for any $N \in \text{Perf } A$, $M \in \text{Perf } A^{\text{op}}$,*

$$(3.3) \quad \sum_n (-1)^n \dim H^n(N \otimes_A M) = \langle \text{Eu}(N), \text{Eu}(M) \rangle.$$

This theorem is an easy corollary of the results of the previous section. Indeed, consider the DG functors:

$$T_N = - \otimes_k N : \text{Perf } k \rightarrow \text{Perf } A, \quad T_M = - \otimes_A M : \text{Perf } A \rightarrow \text{Perf } k,$$

$$T_{N \otimes_A M} = - \otimes_k (N \otimes_A M) : \text{Perf } k \rightarrow \text{Perf } k.$$

Clearly, $T_{N \otimes_A M} = T_M T_N$ and, by Theorem 3.4, we get the equality

$$\text{Eu}(N \otimes_A M) = \langle \text{Eu}(N), \text{Eu}(M) \rangle.$$

What remains is to observe that, for a perfect DG k -module X ,

$$\text{Eu}(X) = \sum_n (-1)^n \dim H^n(X).$$

This latter statement is a corollary of Propositions 3.1 and 3.2, along with the fact that X is homotopy equivalent to $H^\bullet(X)$.

Let us explain how one can compute the right-hand side of (3.3).

First of all, observe that, by Theorem 2.6, the pairing (3.2) induces a pairing on $\text{HH}_\bullet(A) \times \text{HH}_\bullet(A^{\text{op}})$. Let us fix two cycles

$$\sum_a a_0[a_1 | \dots | a_l] \in \mathbf{C}_\bullet(A), \quad \sum_b b_0[b_1 | \dots | b_m] \in \mathbf{C}_\bullet(A^{\text{op}})$$

(\sum indicates that a and b are sums of several terms) and denote by a (resp. b) the corresponding elements in $\text{HH}_\bullet(A)$ (resp. $\text{HH}_\bullet(A^{\text{op}})$). Let us describe $\langle a, b \rangle$ more explicitly.

Consider the composition of DG functors

$$A \otimes A^{\text{op}} \rightarrow \text{Perf}(A \otimes A^{\text{op}}) \xrightarrow{T_\Delta} \text{Perf } k,$$

where $A \otimes A^{\text{op}}$ is viewed as a DG category with one object. Clearly, the unique object of $A \otimes A^{\text{op}}$ gets mapped under this composition to $A \in \text{Perf } k$ and an element $x \otimes y \in A \otimes A^{\text{op}}$, viewed as a morphism in the DG category $A \otimes A^{\text{op}}$, gets mapped to the operator $L(x)R(y) \in \text{End}_k(A)$, where

$$L(x) : c \mapsto xc, \quad R(y) : c \mapsto (-1)^{|c||y|} cy$$

are the operators of left multiplication with x resp. right multiplication with y .

Since the operators of left multiplication commute with operators of right multiplication, we can define a product

$$(3.4) \quad a \wedge b = \sum_{a,b} \pm L(a_0)R(b_0) \text{sh}_{lm}[L(a_1)| \dots | L(a_l)|R(b_1)| \dots | R(b_m)]$$

on $\mathrm{HH}_\bullet(A) \times \mathrm{HH}_\bullet(A^{\mathrm{op}})$ with values in $\mathrm{HH}_\bullet(\mathrm{End}_k(A))$ (the formula for \pm and the definition of sh_{lm} are the same as in (2.2)). Then

$$(3.5) \quad \langle a, b \rangle = \int a \wedge b$$

where \int is defined as follows. Let X be a perfect DG k -module. Then we have an embedding of DG categories⁸ $\mathrm{End}_k(X) \rightarrow \mathrm{Perf}k$ which sends the unique object of the first category to X , viewed as an object of $\mathrm{Perf}k$. Then \int is the map from $\mathrm{HH}_\bullet(\mathrm{End}_k(X))$ to $\mathrm{HH}_\bullet(\mathrm{Perf}k) \simeq k$ induced by this embedding.

Furthermore, let us use the notation $\mathrm{eu}(N)$ to denote the element in $\mathrm{HH}_0(A)$ corresponding to $\mathrm{Eu}(N)$ under the isomorphism $\mathrm{HH}_0(A) \rightarrow \mathrm{HH}_0(\mathrm{Perf}A)$. We are ready to rewrite the right-hand side of (3.3):

$$\langle \mathrm{Eu}(N), \mathrm{Eu}(M) \rangle = \int \mathrm{eu}(N) \wedge \mathrm{eu}(M).$$

It turns out that there are very explicit formulas for \int and eu which will be derived in the next section.

To conclude this section, we will rewrite the Hirzebruch-Riemann-Roch formula in a more conventional form. Namely, we will use (3.3) to derive a formula that expresses the Euler form

$$\chi(M, N) = \sum_n (-1)^n \dim \mathrm{Hom}_{\mathrm{Ho}(\mathrm{Perf}A)}(M, N[n])$$

in terms of the Euler classes of M and N , where M and N are two perfect DG A -modules.

Consider the following (contravariant) DG functor

$$(3.6) \quad D : \mathrm{Mod}A \rightarrow \mathrm{Mod}A^{\mathrm{op}}, \quad M \mapsto DM = \mathrm{Hom}_{\mathrm{Mod}A}(M, A).$$

It is not hard to show that this DG functor preserves perfect modules. Moreover, its square is isomorphic to the identity endofunctor of $\mathrm{Perf}A$ and, thus, D is a quasi-equivalence of the DG categories $(\mathrm{Perf}A)^{\mathrm{op}}$ and $\mathrm{Perf}A^{\mathrm{op}}$. The crucial property of this functor is the following fact: for any perfect DG A -modules there is a natural quasi-isomorphism of complexes

$$N \otimes_A DM \cong \mathrm{Hom}_{\mathrm{Perf}A}(M, N).$$

Thus, the formula (3.3) can be written as follows: for any $N, M \in \mathrm{Perf}A$

$$(3.7) \quad \chi(M, N) = \langle \mathrm{Eu}(N), \mathrm{Eu}(DM) \rangle = \int \mathrm{eu}(N) \wedge \mathrm{eu}(DM).$$

Finally, we notice that $\mathrm{Eu}(DM)$ (and $\mathrm{eu}(DM)$) can be expressed in terms of $\mathrm{Eu}(M)$ (resp. $\mathrm{eu}(M)$). This is based on the following result (see Appendix A):

Proposition 3.6. *For any DG algebra, the formula*

$$(3.8) \quad (a_0[a_1|a_2|\dots|a_n])^\vee = (-1)^{n+\sum_{1 \leq i < j \leq n} |sa_i||sa_j|} a_0[a_n|a_{n-1}|\dots|a_1].$$

defines a quasi-isomorphism $^\vee : \mathbf{C}_\bullet(A) \rightarrow \mathbf{C}_\bullet(A^{\mathrm{op}})$.

⁸For a complex X of vector spaces $\mathrm{End}_k(X)$ stands for the DG algebra $\oplus_n \mathrm{End}_k^n(X)$ where $\mathrm{End}_k^n(X)$ is the subspace of degree n linear maps.

One can generalize the above formulas to the case of an arbitrary DG category to get a quasi-isomorphism ${}^\vee : \mathbf{C}_\bullet(\mathcal{A}) \rightarrow \mathbf{C}_\bullet(\mathcal{A}^{\text{op}})$. In the case $\mathcal{A} = \text{Perf } A$ one can compose it with $\mathbf{C}(D) : \mathbf{C}_\bullet((\text{Perf } A)^{\text{op}}) \rightarrow \mathbf{C}_\bullet(\text{Perf } A^{\text{op}})$ to get a quasi-isomorphism ${}^\vee : \mathbf{C}_\bullet(\text{Perf } A) \rightarrow \mathbf{C}_\bullet(\text{Perf } A^{\text{op}})$. It is immediate that $\text{Eu}(DM) = \text{Eu}(M)^\vee$. It is also true, but is less obvious, that $\text{eu}(DM) = \text{eu}(M)^\vee$. This latter observation follows from the fact that the two quasi-isomorphisms ${}^\vee : \mathbf{C}_\bullet(A) \rightarrow \mathbf{C}_\bullet(A^{\text{op}})$ and ${}^\vee : \mathbf{C}_\bullet(\mathcal{A}) \rightarrow \mathbf{C}_\bullet(\mathcal{A}^{\text{op}})$ agree under the embeddings $A \rightarrow \text{Perf } A$ and $A^{\text{op}} \rightarrow \text{Perf } A^{\text{op}}$.

So here is the Hirzebruch-Riemann-Roch formula in its ultimate form:

$$(3.9) \quad \chi(M, N) = \langle \text{Eu}(N), \text{Eu}(M)^\vee \rangle = \int \text{eu}(N) \wedge \text{eu}(M)^\vee.$$

4. ON THE COMPUTATIONAL ASPECT OF THE HRR THEOREM

4.1. Computing Euler classes. The aim of this section is to explain how to compute the Euler class $\text{eu}(N) \in \text{HH}_0(A)$ of a perfect DG A -module.

The definition of a finitely generated semi-free module we gave in Section 2.2 is convenient for proving theorems but it is not explicit enough for the purposes of this section. A more explicit description was given in [7] and we will begin by recalling it.

Let A be a DG algebra. Let $\text{Free } A$ be the DG subcategory in $\text{Perf } A$ whose objects are finitely generated free DG A -modules, i.e. direct sums of modules of the form

$$A[r] = k[r] \otimes A, \quad r \in \mathbb{Z}.$$

Clearly,

$$\text{Hom}_{\text{Free } A}(A[r], A[s]) = \text{Hom}_{\text{Perf } A}(A[r], A[s]) \simeq A[s - r].$$

The differential on the morphism spaces of the DG category $\text{Free } A$, as well as on the free modules themselves, will be denoted by d_{Free} .

The alternative description of finitely generated semi-free modules is based on the notion of a twisted A -modules. These are objects of a larger DG subcategory $\text{Tw } A \supset \text{Free } A$ in $\text{Perf } A$. Namely, a twisted A -module is a right DG A -module of the form $(\bigoplus_{j=1}^n A[r_j], d_{\text{Free}} + \alpha)$, where $\alpha = (\alpha_{ij})$ is a strictly upper triangular $n \times n$ -matrix of morphisms $\alpha_{ij} \in \text{Hom}_{\text{Free } A}^1(A[r_j], A[r_i])$ satisfying the Maurer-Cartan equation

$$d_{\text{Free}}(\alpha) + \alpha \cdot \alpha = 0.$$

Clearly, the differential d_{Tw} on $\text{Hom}_{\text{Perf } A}((\bigoplus_{j=1}^n A[r_j], d_{\text{Free}} + \alpha), (\bigoplus_{i=1}^m A[s_i], d_{\text{Free}} + \beta))$ is given by the formula

$$d_{\text{Tw}}(f) = d_{\text{Free}}(f) + \beta \cdot f - (-1)^{|f|} f \cdot \alpha.$$

It is not hard to show that any finitely generated semi-free module is isomorphic to a twisted A -modules.

The main result of this section is a formula for the Euler class of a homotopy direct summand of a twisted A -module. Its formulation involves a (super-)trace map⁹ str which we will describe now.

Let N be a DG A -module which is isomorphic to $\bigoplus_{j=1}^n A[r_j]$ as a graded A -module. Fix m homogeneous endomorphisms of N :

$$A', A'', \dots, A^{(m)} \in \text{End}_{\text{Perf } A}(N).$$

Thus, each $A^{(k)}$ is an $n \times n$ -matrix $(e(r_i, r_j) \otimes a_{ij}^{(k)})$ of morphisms

$$e(r_i, r_j) \otimes a_{ij}^{(k)} \in \text{Hom}_{\text{Perf } A}(A[r_j], A[r_i]),$$

where $a_{ij}^{(k)} \in A$ and $e(r_i, r_j) \in \text{Hom}_{\text{Perf } A}(A[r_j], A[r_i])$ is the morphism that sends the generator of $A[r_j]$ to the generator of $A[r_i]$. The endomorphisms give rise to an element $A'[A'' | \dots | A^{(m)}]$ of the Hochschild chain complex of the DG category $\text{Perf } A$. Let us define $\text{str}(A'[A'' | \dots | A^{(m)}]) \in \mathbf{C}_\bullet(A)$ by the formula

$$\text{str}(A'[A'' | \dots | A^{(m)}]) = \sum_{j=1}^n \sum_{i_1, i_2, \dots, i_{m-1}} (-1)^* \cdot a'_{ji_1} [a''_{i_1 i_2} | \dots | a_{i_{m-1} j}^{(m)}],$$

where $*$ is $r_{i_1} + (r_{i_1} - r_j)|a'_{ji_1}| + (r_{i_2} - r_j)|sa''_{i_1 i_2}| + \dots + (r_{i_{m-1}} - r_j)|sa_{i_{m-2} i_{m-1}}^{(m-1)}|$.

Theorem 4.1. *Let $N_\alpha = (\bigoplus_{j=1}^n A[r_j], d_{\text{Free}} + \alpha)$ and L be a homotopy direct summand of N_α corresponding to a homotopy idempotent $\pi : N_\alpha \rightarrow N_\alpha$. Then*

$$\text{eu}(L) = \sum_{l=0}^{n-1} (-1)^l \text{str}(\pi[\underbrace{\alpha | \dots | \alpha}_l])$$

Let us prove the theorem.

Lemma 4.2. *In the above notation, $\text{Eu}(L) = \pi$.*

We have to show that $1_L \in \text{End}_{\text{Perf } A}^0(L)$ and $\pi \in \text{End}_{\text{Perf } A}^0(N_\alpha)$ define the same element of $\text{HH}_0(\text{Perf } A)$. Let us fix some degree 0 closed morphisms $f : N_\alpha \rightarrow L$ and $g : L \rightarrow N_\alpha$ such that

$$fg = 1_L + [d_L, H_L], \quad gf = \pi + [d_{N_\alpha}, H_{N_\alpha}]$$

(see Section 2.1). Then

$$1_L - \pi = b(f[g] + H_{N_\alpha} - H_L).$$

The lemma is proved.

Let $N_\alpha = (\bigoplus_{j=1}^n A[r_j], d_{\text{Free}} + \alpha)$ and π be as before. Let us introduce some new notations.

We will write N_0 to denote the free DG A -module $(\bigoplus_{j=1}^n A[r_j], d_{\text{Free}})$. For an endomorphism

⁹In the case of an associative algebra, this map coincides with the well-known trace map from Section 1.2 of [43].

$f \in \text{End}_{\text{Perf } A}(N_\alpha)$, \tilde{f} (resp. \overrightarrow{f} , \overleftarrow{f}) will stand for f viewed as an element of $\text{End}_{\text{Perf } A}(N_0)$ (resp. $\text{Hom}_{\text{Perf } A}(N_\alpha, N_0)$, $\text{Hom}_{\text{Perf } A}(N_0, N_\alpha)$). For a morphism g we will write g_{ij} (resp. g_{i*} , g_{*j}) for the $n \times n$ -matrix, viewed as a morphism between the same modules, whose ij -th entry (resp. i -th row, j -th column) coincides with that of g and other entries (resp. rows, columns) are 0.

The following lemma is a straightforward consequence of the definition of str :

Lemma 4.3. *One has*

$$\sum_{l=0}^{n-1} (-1)^l \text{str}(\pi[\underbrace{\alpha | \dots | \alpha}_l]) = \sum_{l=0}^{n-1} \sum_{i_0, i_1, \dots, i_l} (-1)^l \text{str}(\tilde{\pi}_{i_0 i_1}[\tilde{\alpha}_{i_1 i_2} | \dots | \tilde{\alpha}_{i_l i_0}])$$

in $\text{HH}_0(\text{Perf } A)$.

The next lemma is less straightforward:

Lemma 4.4. *One has*

$$\pi = \sum_{l=0}^{n-1} \sum_{i_0, i_1, \dots, i_l} (-1)^l \tilde{\pi}_{i_0 i_1}[\tilde{\alpha}_{i_1 i_2} | \dots | \tilde{\alpha}_{i_l i_0}]$$

in $\text{HH}_0(\text{Perf } A)$.

To prove this, pick a large N and apply the differential b to the element

$$\sum_{l=0}^N \sum_{i_0, i_1, \dots, i_l} (-1)^l \overrightarrow{\pi}_{i_0*}[\overleftarrow{1}_{*i_1} | \tilde{\alpha}_{i_1 i_2} | \dots | \tilde{\alpha}_{i_l i_0}].$$

Let us begin by computing the b_0 -component:

$$\begin{aligned} b_0(\overrightarrow{\pi}_{i_0*}[\overleftarrow{1}_{*i_1} | \tilde{\alpha}_{i_1 i_2} | \dots | \tilde{\alpha}_{i_l i_0}]) &= d_{\text{Tw}}(\overrightarrow{\pi}_{i_0*})[\overleftarrow{1}_{*i_1} | \tilde{\alpha}_{i_1 i_2} | \dots | \tilde{\alpha}_{i_l i_0}] \\ &\quad - \overrightarrow{\pi}_{i_0*}[d_{\text{Tw}}(\overleftarrow{1}_{*i_1}) | \tilde{\alpha}_{i_1 i_2} | \dots | \tilde{\alpha}_{i_l i_0}] \\ &\quad + \sum_{m=1}^l \overrightarrow{\pi}_{i_0*}[\overleftarrow{1}_{*i_1} | \tilde{\alpha}_{i_1 i_2} | \dots | d_{\text{Free}}(\tilde{\alpha}_{i_m i_{m+1}}) | \dots | \tilde{\alpha}_{i_l i_0}]. \end{aligned}$$

Recall that π is closed, i.e. $d_{\text{Free}}(\pi) + \alpha\pi - \pi\alpha = 0$. Therefore

$$\begin{aligned} d_{\text{Tw}}(\overrightarrow{\pi}_{i_0*}) &= d_{\text{Free}}(\overrightarrow{\pi}_{i_0*}) - \overrightarrow{\pi}_{i_0*}\alpha = (\overrightarrow{\pi}\alpha)_{i_0*} - (\tilde{\alpha}\overrightarrow{\pi})_{i_0*} - \overrightarrow{\pi}_{i_0*}\alpha \\ &= -(\tilde{\alpha}\overrightarrow{\pi})_{i_0*} = -\sum_{k=1}^n \tilde{\alpha}_{i_0 k} \overrightarrow{\pi}_{k*}. \end{aligned}$$

Furthermore,

$$d_{\text{Tw}}(\overleftarrow{1}_{*i_1}) = \alpha \overleftarrow{1}_{*i_1} = \overleftarrow{\alpha}_{*i_1}, \quad d_{\text{Free}}(\tilde{\alpha}_{i_m i_{m+1}}) = -\sum_{k=1}^n \tilde{\alpha}_{i_m k} \tilde{\alpha}_{k i_{m+1}}.$$

Thus,

$$\begin{aligned} b_0(\vec{\pi}_{i_0*}[\overleftarrow{1}_{*i_1}|\tilde{\alpha}_{i_1i_2}|\dots|\tilde{\alpha}_{i_li_0}]) &= -\sum_{k=1}^n \tilde{\alpha}_{i_0k} \vec{\pi}_{k*}[\overleftarrow{1}_{*i_1}|\tilde{\alpha}_{i_1i_2}|\dots|\tilde{\alpha}_{i_li_0}] \\ &\quad - \vec{\pi}_{i_0*}[\overleftarrow{\alpha}_{*i_1}|\tilde{\alpha}_{i_1i_2}|\dots|\tilde{\alpha}_{i_li_0}] \\ &\quad - \sum_{m=1}^l \sum_{k=1}^n \vec{\pi}_{i_0*}[\overleftarrow{1}_{*i_1}|\tilde{\alpha}_{i_1i_2}|\dots|\tilde{\alpha}_{i_mk}\tilde{\alpha}_{ki_{m+1}}|\dots|\tilde{\alpha}_{i_li_0}]. \end{aligned}$$

Let us compute now the b_1 -component. Clearly, $b_1(\vec{\pi}_{i_0*}[\overleftarrow{1}_{*i_0}]) = \tilde{\pi}_{i_0i_0} - \pi_{i_0i_0}$ and for $l \geq 1$

$$\begin{aligned} b_1(\vec{\pi}_{i_0*}[\overleftarrow{1}_{*i_1}|\tilde{\alpha}_{i_1i_2}|\dots|\tilde{\alpha}_{i_li_0}]) &= \tilde{\pi}_{i_0i_1}[\tilde{\alpha}_{i_1i_2}|\dots|\tilde{\alpha}_{i_li_0}] - \vec{\pi}_{i_0*}[\overleftarrow{\alpha}_{i_1i_2}|\dots|\tilde{\alpha}_{i_li_0}] \\ &\quad - \sum_{m=1}^{l-1} \vec{\pi}_{i_0*}[\overleftarrow{1}_{*i_1}|\tilde{\alpha}_{i_1i_2}|\dots|\tilde{\alpha}_{i_mi_{m+1}}\tilde{\alpha}_{i_{m+1}i_{m+2}}|\dots|\tilde{\alpha}_{i_li_0}] \\ &\quad - \tilde{\alpha}_{i_li_0} \vec{\pi}_{i_0*}[\overleftarrow{1}_{*i_1}|\tilde{\alpha}_{i_1i_2}|\dots|\tilde{\alpha}_{i_{l-1}i_l}]. \end{aligned}$$

To finish the proof of Lemma 4.4, one needs to add the results of the above two computations, take the sum over l and i_0, i_1, \dots, i_l , and observe that the right-hand side of the formula for $b_0(\vec{\pi}_{i_0*}[\overleftarrow{1}_{*i_1}|\tilde{\alpha}_{i_1i_2}|\dots|\tilde{\alpha}_{i_li_0}])$ vanishes for l large enough since α is upper-triangular.

Now Theorem 4.1 follows from the above three lemmas and the following proposition:

Proposition 4.5. *Let N_0 be a free A -module. Then the map*

$$\text{str} : \mathbf{C}_\bullet(\text{End}_{\text{Perf } A}(N_0)) \rightarrow \mathbf{C}_\bullet(A)$$

is a quasi-isomorphism of complexes. Moreover, for any $x \in \text{HH}_\bullet(\text{End}_{\text{Perf } A}(N_0))$ one has $x = \text{str}(x)$ in $\text{HH}_\bullet(\text{Perf } A)$.

The proof of this statement is very similar to the proof of Theorem 1.2.4 from [43] and we will omit it.

4.2. Computing the integral. In Section 3.2 we introduced an “integral”

$$\int : \text{HH}_\bullet(\text{End}_k(X)) \rightarrow \text{HH}_\bullet(\text{Perf } k) \simeq k$$

for any complex of vector spaces X with finite dimensional total cohomology. In this section we will present an explicit formula for this integral based on the results of [22] (see also [51]). This, together with (3.4), will give us an explicit formula for computing the pairing (3.5).

To exclude the trivial case, we will assume that X has non-zero cohomology.

Let us fix a pair of degree 0 maps $p : X \rightarrow H^\bullet(X)$ and $i : H^\bullet(X) \rightarrow X$ that establish the homotopy equivalence between the complex X and its cohomology $H^\bullet(X)$:

$$pi = 1_{H^\bullet(X)}, \quad ip = 1_X - [d_X, H]$$

where $H : X \rightarrow X$ is a degree -1 map.

Here is an explicit formula for the integral:

Theorem 4.6. *The following map is a quasi-isomorphism:*

$$\phi : \mathbf{C}_\bullet(\mathrm{End}_k(X)) \rightarrow k, \quad T_1[T_2 | \dots | T_n] \mapsto \sum_{j=0}^{n-1} \mathrm{str}_{H^\bullet(X)}(\mathcal{F}_n(\tau^j(T_1[T_2 | \dots | T_n]))),$$

where $\mathrm{str}_{H^\bullet(X)}$ is the ordinary super-trace,

$$\tau(T_1[T_2 | \dots | T_n]) = (-1)^{|sT_n|(|sT_1| + \dots + |sT_{n-1}|)} T_n[T_1 | \dots | T_{n-1}],$$

and $\mathcal{F}_n : \mathrm{End}_k(X)^{\otimes n} \rightarrow \mathrm{End}_k(H^\bullet(X))$ is given by

$$\mathcal{F}_n(T_1[T_2 | \dots | T_n]) = pT_1HT_2H \cdot \dots \cdot HT_ni.$$

Furthermore, the induced isomorphism $\mathrm{HH}_\bullet(\mathrm{End}_k(X)) \simeq k$ coincides with \int .

Let us sketch the idea of the proof. That ϕ is a morphism of complexes can be verified by a direct computation. Alternatively, this follows from Lemma 2.4 of [22] and the fact that the collection \mathcal{F}_n , $n = 1, 2, \dots$, gives rise to an A_∞ -morphism from the DG algebra $\mathrm{End}_k(X)$ to the DG algebra $\mathrm{End}_k(H^\bullet(X))$. Moreover, the latter morphism is an A_∞ -quasi-isomorphism, therefore $\mathrm{HH}_\bullet(\mathrm{End}_k(X)) \simeq \mathrm{HH}_\bullet(\mathrm{End}_k(H^\bullet(X))) \simeq k$ which proves that ϕ is a quasi-isomorphism.

It remains to prove that the induced map $\mathrm{HH}_\bullet(\mathrm{End}_k(X)) \rightarrow k$ coincides with \int . Obviously, it suffices to fix a non-zero generator of $\mathrm{HH}_0(\mathrm{End}_k(X))$ and to show that the values of both functionals on this generator coincide. Let us start by describing a generator of $\mathrm{HH}_0(\mathrm{End}_k(X))$.

The endomorphism ip is an idempotent. Let us denote its image by $\mathrm{Harm}^\bullet(X)$. Clearly, $\mathrm{Harm}^\bullet(X)$ is a finite dimensional subspace of X isomorphic to $H^\bullet(X)$. Fix n such that the component $\mathrm{Harm}^n(X)$ is non-zero and let π stand for the projection in $\mathrm{Harm}(X)$ onto this component parallel to other graded components. Then the endomorphism $\Pi = \pi ip \in \mathrm{End}_k^0(X)$ represents a non-zero element of $\mathrm{HH}_0(\mathrm{End}_k(X))$. It is immediate that $\phi(\Pi) = (-1)^n \dim H^n(X)$.

On the other hand, Π and $p\pi i$ define the same element of $\mathrm{HH}_0(\mathrm{Perf}k)$ ($p\pi i$ is just for the projection in $H^\bullet(X)$ onto the component $H^n(X)$ parallel to other graded components). Indeed, $\Pi - p\pi i = \pi ip - p\pi i = b(\pi i[p])$. To finish the proof, observe that the element of $\mathrm{HH}_0(\mathrm{Perf}k)$, defined by $p\pi i$, coincides with the one, defined by $(-1)^n \dim H^n(X) \cdot 1 \in \mathrm{End}_k(k)$.

Let us point out a couple of straightforward corollaries of Theorem 4.6 and formula (3.5).

Let A be a finite dimensional associative algebra. Then its Hochschild homology groups $\mathrm{HH}_\bullet(A)$ are concentrated in non-positive degrees. Therefore, among the pairings $\langle \cdot, \cdot \rangle : \mathrm{HH}_n(A) \times \mathrm{HH}_{-n}(A^{\mathrm{op}}) \rightarrow k$, only the one corresponding to $n = 0$ survives. In this case we have

Corollary 4.7. *For an associative algebra A , the pairing*

$$\langle \cdot, \cdot \rangle : A/[A, A] \times A^{\mathrm{op}}/[A^{\mathrm{op}}, A^{\mathrm{op}}] \rightarrow k$$

is given by

$$\langle a, b \rangle = \text{tr}_A(L(a)R(b)).$$

(In the right-hand side, a and b stand for elements of A and A^{op} , respectively, and in the left-hand side a, b stand for the corresponding classes in the Hochschild homology.)

Let now A be a finite dimensional graded algebra. Since A is equipped with the trivial differential, we can set $H = 0$ in Theorem 4.6 and obtain

Corollary 4.8. *For a graded A , the pairing of two cycles*

$$a = a_0 + \sum a'_0[a'_1] + \sum a''_0[a''_1|a''_2] + \dots \in \mathbf{C}_\bullet(A),$$

$$b = b_0 + \sum b'_0[b'_1] + \sum b''_0[b''_1|b''_2] + \dots \in \mathbf{C}_\bullet(A)$$

is given by

$$\langle a, b \rangle = \text{str}_A(L(a_0)R(b_0)).$$

5. EXAMPLES

5.1. Directed algebras. In this section, we describe how the Hirzebruch-Riemann-Roch formula looks like for a special class of finite dimensional associative algebras.

Let \mathcal{V} be a k -linear category with finite number of objects, say $\{v_s\}_{s \in S}$, and finite dimensional Hom-spaces. Suppose there is a bijection

$$f : \{1, 2, \dots, n\} \rightarrow I$$

such that

$$(5.1) \quad \text{Hom}_{\mathcal{V}}(v_{f(i)}, v_{f(j)}) = \begin{cases} k & i = j \\ 0 & i > j \end{cases}.$$

Of course, f doesn't have to be unique. Let us denote the algebra of this category by $A(\mathcal{V})$:

$$A(\mathcal{V}) = \bigoplus_{s, t \in S} \text{Hom}_{\mathcal{V}}(v_s, v_t).$$

We will call such algebras (as well as the underlying categories) *directed*.

Let us denote the abelian category of finite dimensional right $A(\mathcal{V})$ -modules by $\text{mod} A(\mathcal{V})$. The following simple result is very well known.

Proposition 5.1. *Any module $N \in \text{mod} A(\mathcal{V})$ admits a projective resolution of finite length.*

Let us prove this. Fix a map f as above and denote $1_{v_{f(i)}}$ simply by 1_i . Denote also the projective modules $1_i A(\mathcal{V})$ by P_i . Clearly,

$$\dim \text{Hom}_{\text{mod} A(\mathcal{V})}(P_i, P_j) = \dim \text{Hom}_{\mathcal{V}}(v_{f(i)}, v_{f(j)}).$$

Thus, by (5.1)

$$(5.2) \quad \dim \operatorname{Hom}_{\operatorname{mod} A(\mathcal{V})}(P_i, P_j) = \begin{cases} k & i = j \\ 0 & i > j \end{cases}.$$

Fix $N \in \operatorname{mod} A(\mathcal{V})$. The canonical morphism

$$p : \bigoplus_{i=1}^n \operatorname{Hom}_{\operatorname{mod} A(\mathcal{V})}(P_i, N) \otimes_k P_i \rightarrow N$$

is surjective. The kernel of this morphism satisfies the property

$$\operatorname{Hom}_{\operatorname{mod} A(\mathcal{V})}(P_n, \operatorname{Ker} p) = 0.$$

To see this, apply the functor $\operatorname{Hom}_{\operatorname{mod} A(\mathcal{V})}(P_n, -)$ to the short exact sequence

$$0 \rightarrow \operatorname{Ker} p \rightarrow \bigoplus_{i=1}^n \operatorname{Hom}_{\operatorname{mod} A(\mathcal{V})}(P_i, N) \otimes_k P_i \rightarrow N \rightarrow 0$$

and use the property (5.2).

To finish the proof, apply the same argument to $\operatorname{Ker} p$ instead of N etc.

Observe that $\operatorname{HH}_0(A(\mathcal{V}))$ is spanned by the idempotents 1_{v_s} , $s \in S$ (or rather their classes in the quotient $A(\mathcal{V})/[A(\mathcal{V}), A(\mathcal{V})]$). In terms of these elements, the pairing \langle, \rangle on $\operatorname{HH}_0(A(\mathcal{V})) \times \operatorname{HH}_0(A(\mathcal{V}))^{\operatorname{op}}$ is given by

$$\langle 1_t, 1_s^\vee \rangle = \dim \operatorname{Hom}_{\mathcal{V}}(v_s, v_t).$$

Let us derive the Hirzebruch-Riemann-Roch formula for finite dimensional modules over directed algebras. It is well known and was obtained in [54, Section 2.4].

Let us keep the notations from the proof of Proposition 6.1. Set

$$d_{ij} := \dim \operatorname{Hom}_{\mathcal{V}}(v_{f(i)}, v_{f(j)}).$$

Let $M, N \in \operatorname{mod} A(\mathcal{V})$. As we saw above, M and N admit finite length resolutions by direct sums of the projective modules P_i . Let us fix two such resolutions $P(M)$ and $P(N)$. We know that $\operatorname{eu}(P(M))$, $\operatorname{eu}(P(N))$ are linear combinations of 1_i 's:

$$\operatorname{eu}(P(M)) = \sum_{i=1}^n a_i \cdot 1_i, \quad \operatorname{eu}(P(N)) = \sum_{i=1}^n b_i \cdot 1_i.$$

Since $1_j = \operatorname{eu}(P_j)$, we have

$$\begin{aligned} (\underline{\dim} M)_j &:= \operatorname{Hom}_{\operatorname{mod} A(\mathcal{V})}(P_j, M) = \operatorname{Hom}_{\operatorname{Ho}(\operatorname{Mod} A(\mathcal{V}))}(P_j, P(M)) = \langle \operatorname{eu}(P(M)), 1_j^\vee \rangle \\ &= \sum_{i=1}^n d_{ji} a_i \end{aligned}$$

and similarly $(\underline{\dim} N)_j = \sum_{i=1}^n d_{ji} b_i$. Therefore,

$$\begin{aligned} \sum_l (-1)^l \dim \operatorname{Ext}_{\operatorname{mod} A(\mathcal{V})}^l(M, N) &= \chi(P(M), P(N)) = \langle \operatorname{eu}(P(N)), \operatorname{eu}(P(M))^\vee \rangle \\ &= \sum_{i,j} b_i a_j d_{ji}. \end{aligned}$$

Since $a_j = \sum_k (d^{-1})_{jk}(\underline{\dim} M)_k$, $b_i = \sum_k (d^{-1})_{ik}(\underline{\dim} N)_k$, we get the following generalization of Ringel's formula:

$$(5.3) \quad \sum_l (-1)^l \dim \operatorname{Ext}_{\operatorname{mod} A(\mathcal{V})}^l(M, N) = \sum_{i,j} (\underline{\dim} M)_i (d^{-1})_{ij} (\underline{\dim} N)_j.$$

5.2. Proper noncommutative DG-schemes arising from orbifold singularities.

In this section, we will describe certain proper DG algebras¹⁰ which arise from quotient singularities of the form \mathbb{C}^n/G , where G is a finite group.

Let $V = \mathbb{C}^n$ be a finite dimensional complex vector space and G a finite subgroup of $SL(V) \cong SL_n(\mathbb{C})$. Then G acts on the polynomial algebra $\mathbb{C}[V]$ via $(gf)(x) = f(g^{-1}x)$. The spectrum $X = V/G$ of the algebra $\mathbb{C}[V]^G$ of G -invariant polynomials is a singular affine variety. The central problem in the study of such singular varieties is to construct their “most economical” resolutions, which are called *crepant*: a resolution $\pi : Y \rightarrow X$ is crepant, if π preserves the canonical classes¹¹, i.e. $\pi^*(\omega_X) = \omega_Y$.

The derived McKay correspondence [52, 53, 31, 12] is a program around the following conjecture and various versions thereof:

For any crepant resolution $Y \rightarrow X$, the bounded derived category $D(Y)$ of coherent sheaves on Y is equivalent to the bounded derived category $D^G(V)$ of G -equivariant coherent sheaves on V .

In other words, all crepant resolutions of a fixed singularity are expected to be isomorphic as noncommutative DG-schemes. The conjecture is known to be true for finite subgroups of $SL(2)$ [31] and $SL(3)$ [12] (see also [5] for a result in higher dimensions).

Denote by $D_0^G(V)$ the subcategory in $D^G(V)$ of complexes supported at the origin $0 \in V$ and by $D_0(Y)$ the subcategory in $D(Y)$ of complexes supported at the exceptional fiber $\pi^{-1}(0)$ (in the latter formula 0 stands for the image of the origin of V under the canonical projection $V \rightarrow X$). Then the above equivalence of categories should induce an equivalence between $D_0(Y)$ and $D_0^G(V)$ [12].

The Ext groups between any two objects of $D_0^G(V)$ vanish in all but finitely many degrees and, thus, we are dealing with a proper noncommutative DG-scheme. This scheme is the main subject of the section.

Following [26, Section 6.2], consider the cross-product $\Lambda(V, G)$ of the exterior algebra ΛV and the group algebra of G . In other words, as a vector space $\Lambda(V, G)$ is the tensor product $\Lambda V \otimes \mathbb{C}[G]$. The product of two elements is given by

$$(v \otimes g)(w \otimes h) = (v \wedge g(w)) \otimes gh, \quad v, w \in \Lambda V, g, h \in G.$$

Equip $\Lambda(V, G)$ with the unique grading such that $\deg v = 1$ and $\deg g = 0$ for any $v \in \Lambda V$ and $g \in G$. We will view $\Lambda(V, G)$ as a DG algebra with the trivial differential.

The following conjecture is motivated by [26]:

¹⁰All of them are DG algebras with the trivial differential.

¹¹A crepant resolutions of X , if exists, is a noncompact Calabi-Yau variety since the top-degree form on V is G -invariant and therefore the canonical sheaves of X and Y are trivial.

Conjecture. *There is an equivalence of triangulated categories*

$$D_0^G(V) \cong \mathrm{Ho}(\mathrm{Perf} \Lambda(V, G)).$$

Here is how the conjecture might be proved. The category $D_0^G(V)$ seems to be equivalent to the category $D^b(f.d. \mathbb{C}[V] \rtimes G)$, where $\mathbb{C}[V] \rtimes G$ is the cross-product of the polynomial algebra and the group algebra of G and $f.d. \mathbb{C}[V] \rtimes G$ is the abelian category of finite dimensional graded $\mathbb{C}[V] \rtimes G$ -modules. Every such module admits a finite filtration by simple $\mathbb{C}[V] \rtimes G$ -modules. The latter are the $\mathbb{C}[V] \rtimes G$ -modules obtained from simple $\mathbb{C}[G]$ -modules via “restriction of scalars”

$$\mathbb{C}[V] \rtimes G \rightarrow \mathbb{C}[G], \quad f(x) \otimes g \mapsto f(0)g, \quad f(x) \in \mathbb{C}[V], g \in G.$$

Let us denote the simple $\mathbb{C}[V] \rtimes G$ -module, corresponding to an irreducible representations ρ of G , by S_ρ . Then, using the technique described in [37], we may conclude that $D^b(f.d. \mathbb{C}[V] \rtimes G)$ is equivalent to the category $\mathrm{Ho}(\mathrm{Perf} \mathcal{A})$ for some A_∞ algebra \mathcal{A} with

$$H^\bullet(\mathcal{A}) = \mathrm{Ext}^\bullet(\oplus_\rho S_\rho, \oplus_\rho S_\rho),$$

where the sum in the right-hand side is taken over irreducible representations of G . According to [26, Section 6.2], the algebra $\mathbb{C}[V] \rtimes G$ is quadratic and Koszul, and its Koszul dual is exactly $\Lambda(V, G)$. Then, by [37, Section 2.2], the A_∞ algebra \mathcal{A} is formal. Finally, we expect that $\mathrm{Ext}^\bullet(\oplus_\rho S_\rho, \oplus_\rho S_\rho)$ is Morita equivalent to $\Lambda(V, G)$.

Whether the conjecture is true or not, it is clear that the algebraic triangulated categories of the form $\mathrm{Ho}(\mathrm{Perf} \Lambda(V, G))$ should play a role in the study of the quotient singularities.

Let us compute the pairing $\langle \cdot, \cdot \rangle$ on $\mathrm{HH}_0(\Lambda(V, G)) \times \mathrm{HH}_0(\Lambda(V, G)^{\mathrm{op}})$.

We start by noticing that, in general, the space $\mathrm{HH}_0(\Lambda(V, G))$ is infinite dimensional (this is already so in the simplest case $V = \mathbb{C}$, $G = \{1\}$). However, the pairing $\langle \cdot, \cdot \rangle$ vanishes on a subspace of finite codimension (this follows from Corollary 4.8). In fact, the pairing is determined by its restriction onto the finite dimensional subspace

$$\mathrm{HH}_0(\mathbb{C}[G]) \times \mathrm{HH}_0(\mathbb{C}[G]^{\mathrm{op}}) \subset \mathrm{HH}_0(\Lambda(V, G)) \times \mathrm{HH}_0(\Lambda(V, G)^{\mathrm{op}}).$$

(Here we are using the natural embedding $\mathbb{C}[G] \rightarrow \Lambda(V, G)$ which induces an embedding $\mathrm{HH}_0(\mathbb{C}[G]) \rightarrow \mathrm{HH}_0(\Lambda(V, G))$.) Furthermore, it is well known that $\mathrm{HH}_0(\mathbb{C}[G])$ is spanned by (the homology classes of) the characters of irreducible representations of G . Let us denote the character of an irreducible representation ρ by χ_ρ :

$$\chi_\rho = \sum_g \mathrm{tr}(\rho(g))g.$$

Using basic harmonic analysis on G , it is easy to show that the element $\pi_\rho = \frac{\dim \rho}{|G|} \chi_\rho$ is an idempotent in $\Lambda(V, G)$ (it is nothing but the Euler class of the DG $\Lambda(V, G)$ -module $\pi_\rho \cdot \Lambda(V, G)$). Thus, we have to compute

$$\langle \pi_{\rho_1}, \pi_{\rho_2}^\vee \rangle = \mathrm{str}_{\Lambda(V, G)}(L(\pi_{\rho_1})R(\pi_{\rho_2}))$$

for two irreducible representations ρ_1, ρ_2 .

Let W be the space of some representation of G . Then $W \otimes \mathbb{C}[G]$ carries a natural $\mathbb{C}[G]$ -bimodule structure, defined as follows:

$$g(w \otimes h)k = g(w) \otimes ghk, \quad w \in W, g, h, k \in G.$$

In particular, the graded components $\Lambda^n(V, G) = \Lambda^n V \otimes \mathbb{C}[G]$ of the algebra $\Lambda(V, G)$ are $\mathbb{C}[G]$ -bimodules and we have

$$\begin{aligned} \text{str}_{\Lambda(V, G)}(L(\pi_{\rho_1})R(\pi_{\rho_2})) &= \sum_{n=0}^{\dim V} (-1)^n \text{tr}_{\Lambda^n(V, G)}(L(\pi_{\rho_1})R(\pi_{\rho_2})) \\ &= \sum_{n=0}^{\dim V} (-1)^n \dim(\pi_{\rho_1} \Lambda^n(V, G) \pi_{\rho_2}). \end{aligned}$$

Therefore, we will start by computing $\dim(\pi_{\rho_1}(W \otimes \mathbb{C}[G])\pi_{\rho_2})$ for an arbitrary W .

Let us introduce a matrix d^W of non-negative integers by the following formula:

$$W \otimes \rho = \bigoplus_{\sigma} d_{\sigma\rho}^W \sigma,$$

where ρ and σ run through the set of irreducible representations of G . Let us denote the representation, dual to ρ , by ρ' . Then, as a $\mathbb{C}[G]$ -bimodule

$$W \otimes \mathbb{C}[G] = \bigoplus_{\rho} (W \otimes \rho) \boxtimes \rho' = \bigoplus_{\rho, \sigma} d_{\sigma\rho}^W \sigma \boxtimes \rho'.$$

Thus,

$$\dim(\pi_{\rho_1}(W \otimes \mathbb{C}[G])\pi_{\rho_2}) = \dim \rho_1 \dim \rho_2 d_{\rho_1 \rho_2}^W,$$

which gives us the following formula for $\langle \pi_{\rho_1}, \pi_{\rho_2}^\vee \rangle$:

$$(5.4) \quad \langle \pi_{\rho_1}, \pi_{\rho_2}^\vee \rangle = \dim \rho_1 \dim \rho_2 \sum_{n=0}^{\dim V} (-1)^n d_{\rho_1 \rho_2}^{\Lambda^n V}.$$

6. MORE ON THE PAIRING $\langle \cdot, \cdot \rangle$

6.1. Smooth proper DG algebras. Recall [41] that a DG algebra is said to be (homologically) smooth if there is a perfect right DG $A^{\text{op}} \otimes A$ -module $P(A)$ together with a quasi-isomorphism $P(A) \rightarrow A$ of right DG $A^{\text{op}} \otimes A$ -modules.

To have an example at hand, observe that

Proposition 6.1. *Any directed algebra is smooth.*

Indeed, it is clear that $A(\mathcal{V})^{\text{op}} \otimes A(\mathcal{V}) \cong A(\mathcal{V}^{\text{op}} \otimes \mathcal{V})$. Therefore, by Proposition 5.1, any finite dimensional $A(\mathcal{V})^{\text{op}} \otimes A(\mathcal{V})$ -module admits a finite projective resolution. What remains is to apply this to $A(\mathcal{V})$ and observe that any finite complex of projective bimodules over an associative algebra is a perfect DG bimodule in our sense.

The aim of this section is to prove that the pairing

$$\langle \cdot, \cdot \rangle : \text{HH}_n(\text{Perf } A) \times \text{HH}_{-n}(\text{Perf } A^{\text{op}}) \rightarrow k,$$

is non-degenerate for any proper smooth DG algebra A . The proof is based on the observation that the pairing is inverse to the Euler class $\text{Eu}(A)$ of the A -bimodule A . The author learned about this idea from [42]¹².

Theorem 6.2. *Let A be a proper smooth DG algebra. Then the pairing \langle , \rangle is non-degenerate.*

Indeed, fix a perfect resolution $P(A) \xrightarrow{p} A$ in the category of right DG $A^{\text{op}} \otimes A$ -modules. Then, for any right perfect DG A -module X , we have a morphism

$$1 \otimes p : X \otimes_A P(A) \rightarrow X \otimes_A A \simeq X.$$

By Proposition 2.5, $1 \otimes p$ is a quasi-isomorphism. On the other hand, by Proposition 2.4, both $X \otimes_A P(A)$ and X are perfect and, in particular, homotopically projective. It is well known that a quasi-isomorphism between two homotopically projective modules is actually a homotopy equivalence (see, for instance, the proof of Lemma 10.12.2.2 in [4]). Thus, $1 \otimes p : X \otimes_A P(A) \rightarrow X \otimes_A A \simeq X$ is a homotopy equivalence.

What we have just proved is that the quasi-isomorphism $P(A) \xrightarrow{p} A$ gives rise to a weak homotopy equivalence of the DG functors $T_{P(A)} \rightarrow \text{Id}_{\text{Perf } A}$ where Id stands for the identity endofunctor. Then, as a corollary of Theorem 2.7, we get the following result: the linear map $\text{HH}(T_{P(A)}) : \text{HH}_{\bullet}(\text{Perf } A) \rightarrow \text{HH}_{\bullet}(\text{Perf } A)$ coincides with the identity map. On the other hand, by Theorem 3.4, the map $\text{HH}(T_{P(A)})$ is given by the 'convolution' with $\text{Eu}'(P(A))$, so the convolution with $\text{Eu}'(P(A))$ is the identity map. This proves that the left kernel of the pairing is trivial, i.e. for any n we have an embedding

$$\text{HH}_n(\text{Perf } A) \rightarrow \text{HH}_{-n}(\text{Perf } A^{\text{op}})^*.$$

One of the results of [60] says that the Hochschild homology of an arbitrary proper smooth DG algebra is finite dimensional. Thus, to prove that the right kernel of the pairing is trivial, it is enough to show that $\dim \text{HH}_n(\text{Perf } A) = \dim \text{HH}_{-n}(\text{Perf } A^{\text{op}})$. This can be done by replacing A by A^{op} in the above argument.

Let us point out one interesting corollary of this result¹³:

Corollary 6.3. *If A is a smooth proper associative algebra then*

$$\text{HH}_n(A) = \begin{cases} A/[A, A] & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Indeed, the Hochschild homology of such an algebra is concentrated in non-positive degrees. Thus, by the non-degeneracy of the pairing, the Hochschild homology groups, sitting in negative degrees, have to vanish.

This corollary, together with Proposition 6.1, implies $\text{HH}_n(A(\mathcal{V})) = 0$ for any directed algebra $A(\mathcal{V})$ and any $n \neq 0$. This result was obtained by a different method in [16].

¹²Although [42] is still in preparation, the argument with the Euler class of the "diagonal" is already well known among the experts [30].

¹³If k is perfect, this result also follows from Proposition 2.5 of [32] and Morita invariance of the Hochschild homology.

Another application of the corollary is related to the so-called noncommutative Hodge-to-de Rham degeneration conjecture. Roughly speaking, the conjecture claims that the B -operator $B : \mathrm{HH}_\bullet(A) \rightarrow \mathrm{HH}_{\bullet-1}(A)$ (see [23, 67]) vanishes whenever A is proper and smooth. It was formulated, in a stronger form, by M. Kontsevich and Y. Soibelman [41] and proved, in the partial case of DG algebras concentrated in non-negative degrees, by D. Kaledin [30]. The above corollary implies the conjecture in the case of algebras with the trivial differential and grading.

6.2. Relation to Topological Field Theories. This section is devoted to yet another application of our results. Namely, we will discuss the relevance of the pairing (3.5) to the Topological Field Theories (TFT's) constructed in [18, 41].

To begin with, let us recall that by a *trace* on a DG algebra A one understands a (homogeneous) functional $\tau : A \rightarrow k$ such that

$$\tau(da) = 0, \quad \tau([a, b]) = 0, \quad a, b \in A.$$

Let A be a proper DG algebra. Suppose the algebra possesses a degree $-d$ trace τ satisfying the following condition: the induced degree $-d$ pairing

$$\mathrm{H}^\bullet(A) \times \mathrm{H}^\bullet(A) \rightarrow k, \quad (a, b) \mapsto \tau(ab)$$

is non-degenerate. Then the pair (A, τ) is called a d -dimensional (compact) Calabi-Yau DG algebra [41]¹⁴. Sometimes, we will write A instead of (A, τ) .

Observe that the algebra $\Lambda(V, G)$, we studied in Section 5.2, carries a natural structure of a $\dim V$ -dimensional CY DG algebra. Namely, fix a non-zero element $\omega \in \Lambda^{\dim V} V$ and set [26]:

$$\tau_\omega(v \otimes g) = \begin{cases} 0 & v \in \Lambda^n V, n < \dim V \\ \delta_{1g} & v = \omega \end{cases}.$$

Before we proceed any further, we would like to mention that there is a different class of CY algebras whose theory is now being actively developed [25]. These latter CY algebras are noncommutative analogs of *noncompact* smooth CY varieties (a good example of such an algebra is the cross-product $\mathbb{C}[V] \rtimes G$ we mentioned in Section 5.2).

A d -dimensional TFT is defined as follows (we refer to [18, 41] for details). Let $\mathcal{M}(n, m) = \bigcup_{g \geq 0} \mathcal{M}_g(n, m)$ denote the moduli space of Riemann surfaces with n incoming and m outgoing boundaries (g denotes the genus). Fix a graded vector space \mathbf{H}_\bullet . By definition, \mathbf{H}_\bullet carries a structure of a d -dimensional TFT if one has a collection of linear maps

$$(6.1) \quad \mathbf{H}_\bullet(\mathcal{M}(n, m)) \otimes \mathbf{H}_\bullet^{\otimes n} \rightarrow \mathbf{H}_\bullet^{\otimes m}, \quad n \geq 1, m \geq 0$$

(here \mathbf{H}_\bullet in the left-hand side denotes the singular homology) satisfying the following conditions:

(1) the maps are compatible with the operation

$$\mathcal{M}(m, l) \times \mathcal{M}(n, m) \rightarrow \mathcal{M}(n, l)$$

¹⁴Actually, the authors of [18, 41] work with CY A_∞ algebras and categories.

of gluing of two surfaces along the boundary components and the operation

$$\mathcal{M}(n, m) \times \mathcal{M}(p, q) \rightarrow \mathcal{M}(n + p, m + q)$$

of taking the disjoint union of surfaces;

(2) elements of $H_\bullet(\mathcal{M}_g(n, m))$ act by operators of degree $d(2 - 2g - n - m)$.

One has [18, 41]:

*For any d -dimensional CY DG algebra A the Hochschild homology $HH_\bullet(A)$ carries a canonical structure of d -dimensional TFT.*¹⁵

One immediate consequence of this result is that there is a natural degree 0 pairing – let us denote it by $\langle \cdot, \cdot \rangle_\tau$ – on the Hochschild homology of a d -dimensional CY DG algebra, given by a generator of $H_0(\mathcal{M}_0(2, 0))$. The following conjecture relates this pairing to the one constructed in the present work¹⁶:

Conjecture. *For any CY DG algebra A , the pairing $\langle \cdot, \cdot \rangle_\tau$ coincides with the pairing (3.5), i.e. for any $a, b \in HH_\bullet(A)$*

$$(6.2) \quad \langle a, b \rangle_\tau = \langle a, b^\vee \rangle,$$

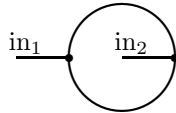
where $^\vee$ is the isomorphism $HH_\bullet(A) \rightarrow HH_\bullet(A^{\text{op}})$ defined by (3.8).

We note that this conjecture, together with Theorem 6.2, would imply the following result conjectured in [41, Section 11.6]:

Corollary. *For any smooth CY DG algebra A , the pairing $\langle \cdot, \cdot \rangle_\tau$ is non-degenerate.*

To present a piece of evidence in favor of the conjecture, let us prove it in the case of an associative Calabi-Yau algebra, when the grading and the differential are both trivial. Observe that such a CY algebra is nothing but a symmetric Frobenius algebra [38].

To compute the left-hand side of (6.2), we will use an explicit description of the action (6.1) based on graphs [41, Section 11.6]. In the language of [41], the pairing $\langle \cdot, \cdot \rangle_\tau$ corresponds to the following graph:



Let us fix a symmetric Frobenius algebra $A = (A, \tau)$. Since A is finite dimensional and the bilinear form $\tau(ab)$ is non-degenerate, there exists a unique element

$$\Phi = \sum_k \phi'_k \otimes \phi''_k \otimes \phi'''_k \in A \otimes A \otimes A$$

satisfying the property

$$(6.3) \quad \tau(abc) = \sum_k \tau(a\phi'_k) \tau(b\phi''_k) \tau(c\phi'''_k)$$

¹⁵In fact, a much stronger result is obtained in [18, 41], namely, that the action (6.1) exists on the level of complexes that compute the singular homology of the moduli spaces and the Hochschild homology of the algebra.

¹⁶This conjecture was suggested to the author by Y. Soibelman and K. Costello.

for every $a, b, c \in A$. Notice that Φ is cyclically symmetric because $\tau(ab)$ is symmetric. According to [41], $\langle a, b \rangle_\tau$ can be computed by means of the above graph as follows: attach a to the vertex marked in_1 and b to the vertex marked in_2 ; attach two copies of the tensor Φ to the remaining two vertices; contract all the tensors along all four edges of the graph, using the pairing $a \times b \mapsto \tau(ab)$. Here is the result:

$$\langle a, b \rangle_\tau = \sum_{k,l} \tau(a\phi'_k) \tau(b\phi'_l) \tau(\phi''_k \phi''_l) \tau(\phi'''_k \phi'''_l).$$

By (6.3), the latter formula can be simplified as follows:

$$\langle a, b \rangle_\tau = \sum_k \tau(a\phi'_k) \tau(b\phi''_k \phi'''_k).$$

To simplify the formula further, consider the unique symmetric element

$$\gamma = \sum_i \gamma'_i \otimes \gamma''_i \in A \otimes A$$

satisfying the property

$$(6.4) \quad a = \sum_i \gamma'_i \tau(\gamma''_i a) = \sum_i \tau(a\gamma'_i) \gamma''_i$$

for every $a \in A$. Then it is easy to see that $\Phi = \sum_{i,j} \gamma'_i \otimes \gamma'_j \gamma''_i \otimes \gamma''_j$. Indeed,

$$\begin{aligned} \sum_{i,j} \tau(a\gamma'_i) \tau(b\gamma'_j \gamma''_i) \tau(c\gamma''_j) &= \sum_i \tau(a\gamma'_i) \tau(b \sum_j \gamma'_j \tau(c\gamma''_j) \gamma''_i) = \sum_i \tau(a\gamma'_i) \tau(bc\gamma''_i) \\ &= \tau(a \sum_i \gamma'_i \tau(bc\gamma''_i)) = \tau(abc). \end{aligned}$$

Thus,

$$\begin{aligned} \langle a, b \rangle_\tau &= \sum_k \tau(a\phi'_k) \tau(b\phi''_k \phi'''_k) = \sum_{i,j} \tau(a\gamma'_i) \tau(b\gamma'_j \gamma''_i \gamma''_j) = \sum_j \tau(b\gamma'_j \sum_i \tau(a\gamma'_i) \gamma''_i \gamma''_j) \\ &= \sum_j \tau(b\gamma'_j a\gamma''_j). \end{aligned}$$

Since γ is symmetric, we arrive at the following formula

$$\langle a, b \rangle_\tau = \sum_i \tau(a\gamma'_i b\gamma''_i).$$

By Corollary 4.7, we have $\langle a, b^\vee \rangle = \text{tr}_A(L(a)R(b))$. Thus, for a symmetric Frobenius algebra, the above conjecture boils down to the following identity:

$$\sum_i \tau(a\gamma'_i b\gamma''_i) = \text{tr}_A(L(a)R(b)), \quad a, b \in A.$$

To prove it, we observe that under the canonical isomorphism $\text{End}_k(A) \cong A \otimes A^*$ the operators $L(a)$, $R(b)$ get mapped to the elements

$$\sum_i a\gamma'_i \otimes \tau(\gamma''_i \cdot -), \quad \sum_j \gamma'_j b \otimes \tau(\gamma''_j \cdot -),$$

respectively (this follows from the definition (6.4) of γ). Therefore,

$$\begin{aligned} \mathrm{tr}_A(L(a)R(b)) &= \sum_{i,j} \tau(\gamma_j'' a \gamma_i') \tau(\gamma_i'' \gamma_j' b) = \sum_i \tau(\gamma_i'' \gamma_j' \sum_j \tau(\gamma_j'' a \gamma_i') b) \\ &= \sum_i \tau(\gamma_i'' a \gamma_i' b) = \sum_i \tau(a \gamma_i' b \gamma_i''), \end{aligned}$$

which finishes the proof.

The same proof should work for graded CY DG algebras with the trivial differential.

APPENDIX A. PROOF OF PROPOSITION 3.6

Clearly, the morphism (3.8) is invertible. We have to show that it commutes with the differentials. It is obvious that $^\vee$ respects the first differential b_0 as its definition doesn't involve multiplication. Let us show by a direct computation that $^\vee$ commutes with the second differential b_1 . Let us denote the multiplication in A^{op} by $*$. To simplify computations, we will also use the notations $\xi_i = |a_0| + |sa_n| + |sa_{n-1}| + \dots + |sa_{i+1}|$ and $f(a_1, a_2, \dots, a_n) = \sum_{1 \leq i < j \leq n} |sa_i| |sa_j|$. One has:

$$\begin{aligned} b_1((a_0[a_1|a_2| \dots |a_n])^\vee) &= (-1)^{n+f(a_1, a_2, \dots, a_n)} b_1(a_0[a_n|a_{n-1}| \dots |a_1]) \\ &= (-1)^{n+f(a_1, a_2, \dots, a_n)} ((-1)^{|a_0|} a_0 * a_n[a_{n-1}| \dots |a_1] \\ &\quad + \sum_{i=1}^{n-1} (-1)^{\xi_i} a_0[a_n|a_{n-1}| \dots |a_{i+1} * a_i| \dots |a_1] \\ &\quad - (-1)^{\xi_1(|a_1|+1)} a_1 * a_0[a_n|a_{n-1}| \dots |a_2]) \\ &= (-1)^{n+f(a_1, a_2, \dots, a_n)} ((-1)^{|a_0|+|a_0||a_n|} a_n a_0[a_{n-1}| \dots |a_1] \\ &\quad + \sum_{i=1}^{n-1} (-1)^{\xi_i+|a_{i+1}||a_i|} a_0[a_n|a_{n-1}| \dots |a_i a_{i+1}| \dots |a_1] \\ &\quad - (-1)^{\xi_1(|a_1|+1)+|a_1||a_0|} a_0 a_1[a_n|a_{n-1}| \dots |a_2]) \end{aligned}$$

On the other hand,

$$\begin{aligned} (b_1(a_0[a_1|a_2| \dots |a_n]))^\vee &= (-1)^{|a_0|} (a_0 a_1[a_2| \dots |a_n])^\vee \\ &\quad + \sum_{i=1}^{n-1} (-1)^{\eta_i} (a_0[a_1|a_2| \dots |a_i a_{i+1}| \dots |a_n])^\vee \\ &\quad - (-1)^{\eta_{n-1}(|a_n|+1)} (a_n a_0[a_1|a_2| \dots |a_{n-1}])^\vee \\ &= (-1)^{|a_0|} (-1)^{n-1+f(a_2, \dots, a_n)} a_0 a_1[a_n| \dots |a_2] \\ &\quad + \sum_{i=1}^{n-1} (-1)^{\eta_i} (-1)^{n-1+f(a_1, a_2, \dots, a_i a_{i+1}, \dots, a_n)} a_0[a_n|a_{n-1}| \dots |a_i a_{i+1}| \dots |a_1] \\ &\quad - (-1)^{\eta_{n-1}(|a_n|+1)} (-1)^{n-1+f(a_1, a_2, \dots, a_{n-1})} a_n a_0[a_{n-1}|a_{n-2}| \dots |a_1] \end{aligned}$$

What remains is to compare the signs, i.e. to show that

$$(-1)^{f(a_1, a_2, \dots, a_n)} (-1)^{|a_0|+|a_0||a_n|} = (-1)^{\eta_{n-1}(|a_n|+1)} (-1)^{f(a_1, a_2, \dots, a_{n-1})},$$

$$(-1)^{f(a_1, a_2, \dots, a_n)} (-1)^{\xi_i + |a_{i+1}| |a_i|} = -(-1)^{\eta_i} (-1)^{f(a_1, a_2, \dots, a_i a_{i+1}, \dots, a_n)},$$

$$(-1)^{f(a_1, a_2, \dots, a_n)} (-1)^{\xi_1(|a_1|+1) + |a_1| |a_0|} = (-1)^{|a_0|} (-1)^{f(a_2, \dots, a_n)},$$

which is an easy computation.

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